pfssoln6.tex

SOLUTIONS 6 22.11.2012

Q1. (i) $EM(t) = 1$, as $N(0,1)$ has MGF $e^{\frac{1}{2}s^2}$ and $N(0,t) =_d \sqrt{t}.N(0,1)$.

$$\begin{aligned} E[M_t|\mathcal{F}_u] &= E[\exp\{s[B_u + (B_t - B_u)] - \tfrac{1}{2}ts^2\}|\mathcal{F}_u] \\ &= \exp\{sB_u - \tfrac{1}{2}us^2\}.E[\exp\{s(B_t - B_u) - \tfrac{1}{2}(t-u)s^2\}|\mathcal{F}_u], \end{aligned}$$

taking out what is known. The first term on RHS is M_u. Using the Strong Markov Property for BM to start afresh at time u, the second is $E[M_{t-u}]$, which is 1 by above. So M is a mg.
(ii) The stopping time T_n is bounded, so Doob's Stopping Time Principle gives $E[M(T_n)] = 1$:

$$\begin{aligned} 1 &= E\exp\{sB(T_n) - T_n.\frac{1}{2}s^2\} \\ &= E[\exp\{sB(n) - n.\frac{1}{2}s^2\}I(\tau > n)] + E[\exp\{sB(\tau_t) - \tau_t.\frac{1}{2}s^2\}I(\tau \le n)]. \end{aligned}$$

On $\tau > n$, $B(n) < t$, so the first term on RHS is at most

$$\exp\{st - n.\frac{1}{2}s^2\}.P(\tau > n) \le e^{st}.P(\tau > n) \to 0 \qquad (n \to \infty).$$

Letting $n \to \infty$, $1 = E[\exp\{st - \tau_t.\frac{1}{2}s^2\}.I(\tau_t < \infty)]$. But $\tau_t < \infty$ a.s. (otherwise B_u would lie below t for all u, so would have support bounded above; but each B_u is normal, so has support the whole line). So

$$1 = E[\exp\{st - \tau_t.\frac{1}{2}s^2\}]: \quad E[\exp\{-\tau_t.\frac{1}{2}s^2\}] = e^{-st}, \quad E[\exp\{-s\tau_t\}] = e^{-t\sqrt{2s}}.$$

(iii) The first-passage process τ is non-decreasing, as it takes longer to reach a higher level. Using the Strong Markov Property at time τ_t shows that the further time to first passage to level $t+u$ is independent of $\mathcal{F}_t$, and so of τ_t; this says that the process τ has independent increments. This further time has the same distribution as τ_u, by the stationary-increments property of BM; so τ has stationary increments. so τ is a non-decreasing Lévy process, i.e. a subordinator.
(iv) By (iii), $E\exp\{-s\tau_{ct}\} = \exp\{-t.c\sqrt{2s}\} = \exp\{-t.\sqrt{2sc^2}\} = E\exp\{-sc^2\tau_t\}$.

Comparing, $c^2 \tau_t =_d \tau_{ct}$: $\tau_t =_d \tau_{ct}/c^2$.

Q2.
$$\phi(s) = \frac{1}{\sqrt{2\pi}} . \int_0^\infty \exp(-sx - \frac{1}{2x}) . \frac{dx}{x^{3/2}}.$$
Differentiate under the integral sign (as we may, the integrand being monotone in s – we quote this):
$$\phi'(s) = -\frac{1}{\sqrt{2\pi}} . \int_0^\infty \exp(-sx - \frac{1}{2x}) . \frac{dx}{\sqrt{x}}.$$
The change of variable suggested interchanges the two terms in the exponential. It reverses the limits, and (check)
$$\frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{2s}} . \frac{du}{u^{3/2}}.$$
This gives
$$\phi'(s) = -\frac{1}{\sqrt{2s}} . \phi(s) : \qquad \frac{\phi'(s)}{\phi(s)} = -\frac{1}{\sqrt{2s}}.$$
Integrate: $\log \phi(s) = -\sqrt{2s} + c$, $\phi(s) = ce^{-\sqrt{2s}}$. But $\phi(0) = \int f = 1$, so $c = 1$ and $\phi(s) = e^{-\sqrt{2s}}$. //
Note. This is *Lévy's density* – a rare case (with the normal and Cauchy) where one can find a stable density explicitly.

Q3. Adding independent random variables multiplies Laplace transforms (as with CFs – from the Multiplication Theorem), so $X_1 + \ldots + X_n$ has Laplace transform $[\phi(s)]^n = e^{-n\sqrt{2s}}$. Replacing s by s/n^2, $X_1 + \ldots + X_n)/n^2$ has Laplace transform $\phi(s) = e^{-\sqrt{2s}}$, the Laplace transform of X. So $(X_1 + \ldots + X_n)/n^2$ has the same distribution as X, as required.

This does not contradict the SLLN, as X has infinite mean.

Q4. $| \int_B f_n - \int_B f | = | \int_B (f_n - f| \leq \int_B |f_n - f|$. Taking sups over B proves the inequality. Next, with $a \wedge b := \min(a, b)$, $|f_n - f| = f_n + f - 2f_n \wedge f$ (check). Integrate: $\int f_n = 1$, $\int f = 1$ as these are densities. As $0 \leq f_n \wedge f \leq f$, integrable, dominated convergence gives $\int f_n \wedge f \to \int f = 1$. So the integral of RHS $\to$ 1+1-2 = 0. So the integral of LHS $\to$ 0 also. // NHB