pfssoln8.tex

**SOLUTIONS 8 6.12.2012**

Q1. Take $\mathcal{C}$ the trivial $\sigma$-field $\{\emptyset, \Omega\}$. This contains no information, so an expectation conditioning on it is the same as an unconditional expectation. The first form of the tower property now gives

$$E[E[X|\mathcal{B}]\ |\{\emptyset, \Omega\}] = E[X|\{\emptyset, \Omega\}] = E[X].$$

Q2. Recall $varX := E[(X - EX)^2]$. Expanding the square,

$$varX = E[X^2 - 2X.(EX) + (EX)^2] = E(X^2) - 2(EX)(EX) + (EX)^2 = E(X^2) - (EX)^2.$$

Conditional variances can be defined in the same way. Recall that $E(Y|X)$ is constant when $X$ is known ($= x$, say), so can be taken outside an expectation over $X$, $E_X$ say. Then

$$var(Y|X) := E(Y^2|X) - [E(Y|X)]^2.$$

Take expectations of both sides over $X$:

$$E_X var(Y|X) = E_X[E(Y^2|X)] - E_X[E(Y|X)]^2.$$

Now $E_X[E(Y^2|X)] = E(Y^2)$, by the Conditional Mean Formula, so the right is, adding and subtracting $(EY)^2$,

$$\{E(Y^2) - (EY)^2\} - \{E_X[E(Y|X)]^2 - (EY)^2\}.$$

The first term is $varY$, by above. Since $E(Y|X)$ has $E_X$-mean $EY$, the second term is $var_X E(Y|X)$, the variance (over $X$) of the random variable $E[Y|X]$ (random because $X$ is). Combining, the result follows.
*Interpretation.* $varY$ = total variability in $Y$,
$E_X var(Y|X)$ = variability in $Y$ not accounted for by knowledge of $X$,
$var_X E(Y|X)$ = variability in $Y$ accounted for by knowledge of $X$.

Q3. (i)

$$\psi(t) = E[e^{itY}] \quad = \quad E[\exp\{it(X_1 + \ldots + X_N)\}]$$

$$
\begin{aligned}
&= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n].P(N = n) \\
&= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1 + \ldots + X_n)\}] \\
&= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n \\
&= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\
&= \exp\{-\lambda(1 - \phi(t))\}.
\end{aligned}
$$

Differentiate:

$$\psi'(t) = \psi(t).\lambda\phi'(t),$$
$$\psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives

$$EY = \lambda\mu.$$

Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2].$$

(ii) Given $N$, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As $N$ is Poisson with parameter $\lambda$, $N$ has mean $\lambda$ and variance $\lambda$. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2].$$

NHB