pfssoln9.tex

SOLUTION 9 13.12.2012

Q1. $u_0 = 1$ by definition; for $n \geq 1$,

$$u_n = P(\text{in } 0 \text{ at time } n) = \sum_{k=1}^{n} P(\text{in } 0 \text{ at time } k \text{ for the first time and at time } n)$$

$$= \sum_{k=1}^{n} P(\text{in } 0 \text{ at time } n \mid \text{in } 0 \text{ at time } k \text{ for the first time}) P(\text{in } 0 \text{ at time } k \text{ for the first time})$$

$$= \sum_{k=1}^{n} u_{n-k} f_k,$$

using the Markov property to restart the chain from 0 at time k. So the GF is

$$U(s) := \sum_{n=0}^{\infty} u_n s^n = 1 + \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} u_{n-k} f_k.$$

Put $j := n - k$; as $1 \leq k \leq n < \infty$, the new limits on summation are $0 \leq j < \infty, 1 \leq k \leq \infty$. We obtain

$$U(s) = 1 + \sum_{j=0}^{\infty} u_j s^j \sum_{k=1}^{\infty} f_k s^k = 1 + U(s)F(s),$$

so $U(s) - U(s)F(s) = 1$:

$$U(s) = 1/(1 - F(s)).$$

Q2.

$$p_{i,i-1} = \frac{i}{d}, \qquad p_{i,i+1} = \frac{d-i}{d}, \qquad \pi_i = 2^{-d} \binom{d}{i}.$$

Now

$$\binom{d}{j-1}(d-j+1) = \frac{d!}{(j-1)!(d-j+1)!}.(d-j+1) = \frac{d!}{(j-1)!(d-j)!} = \binom{d}{j}.j, \qquad (i)$$

$$\binom{d}{j+1}(j+1) = \frac{d!}{(j+1)!(d-j-1)!}(j+1) = \frac{d!}{j!(d-j-1)!} = \binom{d}{j}.(d-j). \qquad (ii)$$

So by (i) and (ii),

$$\begin{aligned}(\pi P)_j &= \sum_i \pi_i p_{ij} \\ &= \pi_{j-1}p_{j-1,j} + \pi_{j+1}p_{j+1,j} \\ &= 2^{-d}\binom{d}{j-1}\frac{(d-j+1)}{d} + 2^{-d}\binom{d}{j+1}\frac{(j+1)}{d} \\ &= \frac{2^{-d}}{d}\binom{d}{j}\{j+(d-j)\} = 2^{-d}\binom{d}{j} \\ &= \pi_j.\end{aligned}$$

So $\pi P = \pi$, and π is invariant, as required. //

Q3. With π as in Q1,

$$\pi_i p_{i,i+1} = 2^{-d}\binom{d}{i}.\frac{d-i}{d} = 2^{-d}\frac{d!}{(d-i)!i!}.\frac{d-i}{d} = 2^{-d}\binom{d-1}{i},$$

and similarly

$$\pi_{i+1}p_{i+1,i} = 2^{-d}\binom{d-1}{i},$$

proving detailed balance, and so reversibility. Assuming reversibility, we can use detailed balance to calculate the invariant distribution:

$$i=0: \qquad \pi_1 = \pi_0\frac{p_{01}}{p_{10}} = \frac{\pi_0}{\frac{1}{d}}.$$

$$i=1: \qquad \pi_2 = \pi_1\frac{p_{12}}{p_{21}} = \frac{\pi_0}{\frac{1}{d}}.\frac{1-\frac{1}{d}}{\frac{2}{d}}, \ldots,$$

$$\pi_i = \frac{\pi_0}{\frac{1}{d}}.\frac{1-\frac{1}{d}}{\frac{2}{d}}.\ldots.\frac{1-\frac{i-1}{d}}{\frac{i}{d}} = \pi_0.\frac{d(d-1)\ldots(d-i+1)}{1.2\ldots.i} = \pi_0\binom{d}{i}.$$

Then $\sum_i \pi_i = 1$ gives

$$\pi_0\sum_i\binom{d}{i} = \pi_0.2^d = 1, \qquad \pi_0 = 2^{-d}, \qquad \pi_i = 2^{-d}\binom{d}{i},$$

as before.

NHB