PROBABILITY FOR STATISTICS: EXAMINATION, 2012-13

Q1. (i) Show that if x is a (column) vector with components x_i independent and standard normal ($N(0,1)$), and

$$y := Ox$$

with O an orthogonal matrix, then the components y_i of y are again independent $N(0,1)$.
(ii) Describe how to construct a *Helmert matrix* – an orthogonal matrix with first row

$$e_1 = (1/\sqrt{n}, \ldots, 1/\sqrt{n}).$$

(iii) For a sample $X_1, \ldots, X_n$ from a normal population $N(\mu, \sigma^2)$, show that the sample mean $\bar{X}$ and the sample variance S^2 are independent.

Q2. (i) Define the moment-generating function (MGF) $M(t) = M_X(t)$ of a random variable X.
(ii) Show that for X, Y independent, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.
(iii) Find the MGF of the standard normal distribution $N(0,1)$. Hence or otherwise find the MGF of the general normal distribution $N(\mu, \sigma^2)$.
(iv) If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ are independent, find the MFG and distribution of $X + Y$.
(v) Define the characteristic function (CF). State briefly, without proofs, the corresponding results with the CF rather than the MGF here.

Q3. For N Poisson distributed with parameter λ and $X_1, X_2, \ldots$ independent of each other and of N, each with distribution F with mean μ, variance σ^2 and characteristic function $\phi(t)$, show that the compound Poisson distribution of

$$Y := X_1 + \ldots + X_N$$

has
(i) characteristic function $\psi(t) = \exp\{-\lambda(1 - \phi(t))\}$,
(ii) mean $\lambda\mu$,
(iii) variance $\lambda E[X^2]$.

Q4. In the Markov chain given by the Ehrenfest urn, d balls are divided between two urns, I and II. At each stage, a ball is picked at random, and transferred to the other urn. The state of the process is the number of balls

in urn I.
(i) Write down the transition probabilities p_{ij}.
(ii) Show that the Ehrenfest urn has no limit distribution, but has as an invariant distribution the *binomial distribution* $\pi = (\pi_j)$, where

$$\pi_j = 2^{-d}\binom{d}{j}.$$

(iii) We say that the chain satisfies the *detailed balance* condition if there exists a distribution $\pi = (\pi_j)$ with $\pi_j > 0$ for all j and

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \text{for all } i,\ j. \tag{DB}$$

The chain is *reversible* if its probabilistic structure is invariant under time-reversal. Show that the Ehrenfest urn is reversible, and derive its invariant distribution from detailed balance. (You may assume that reversibility is equivalent to detailed balance, and that then the π in (DB) is the unique invariant distribution.)

N. H. Bingham