# PROBABILITY FOR STATISTICS: EXAMINATION SOLUTIONS, 2012-13

Q1. (i) The joint density of the $x_i$ is

$$f(x) = (2\pi)^{-\frac{1}{2}n} \prod_{i=1}^{n} \exp\{-\frac{1}{2}x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\sum_1^n x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\|x\|^2\}.$$

The Jacobian of the change of variable is the determinant $|O|$, which is 1 as $O$ is orthogonal (= length-preserving), and $\|y\| = \|x\|$, again by orthogonality. So the joint density of the $y_i$ is

$$g(y) = (2\pi)^{-\frac{1}{2}n} \exp\{-\|y\|^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\sum_1^n y_i^2\},$$

which says that the $y_i$ are iid $N(0,1)$. **[6]**

(ii) The condition for a matrix $O$ to be orthogonal is that the rows are of length 1 and orthogonal vectors. Take the first row as $e_1$, and use Gram-Schmidt orthogonalisation to find $e_2$ orthogonal to $e_1$, then $e_3$ orthogonal to $e_1, e_2$ etc. The $e_i$ form the rows of an orthogonal matrix with first row $e_1$. **[6]**

(iii) Put $Z_i := (X_i - \mu)/\sigma$, $Z := (Z_1, \ldots, Z_n)^T$; then the $Z_i$ are iid $N(0,1)$,

$$\bar{Z} = (\bar{X} - \mu)/\sigma, \qquad nS^2/\sigma^2 = \sum_1^n (Z_i - \bar{Z})^2.$$

Also

$$\sum_1^n Z_i^2 = \sum_1^n (Z_i - \bar{Z})^2 + n\bar{Z}^2,$$

since $\sum_1^n Z_i = n\bar{Z}$. The terms on the right above are quadratic forms, with matrices $A$, $B$ say, so we can write

$$\sum_1^n Z_i^2 = Z^T A Z + Z^T B Z. \qquad \textbf{[6]}$$

Put $W := PZ$ with $P$ a Helmert transformation with first row $(1, \ldots, 1)/\sqrt{n}$:

$$W_1 = \frac{1}{\sqrt{n}} \sum_1^n Z_i = \sqrt{n}\bar{Z}; \qquad W_1^2 = n\bar{Z}^2 = Z^T B Z.$$

So by above,

$$Z^T A Z = \sum_1^n (Z_i - \bar{Z})^2 = nS^2/\sigma^2, = \sum_2^n W_i^2,$$

as $\sum_1^n Z_i^2 = \sum_1^n W_i^2$. But the $W_i$ are independent (by the orthogonality of $P$), so $W_1$ is independent of $W_2, \ldots, W_n$. So $W_1^2$ is independent of $\sum_2^n W_i^2$. So $nS^2/\sigma^2$ is independent of $n(\bar{X} - \mu)^2/\sigma^2$, so $S^2$ is independent of $\bar{X}$. **[7]**
[Seen – Problems]

Q2. (i) For a random variable $X$ all of whose moments $\mu_n := E[X^n]$ exist, the *moment-generating function (MGF)* of $X$ is, for $t$ real,

$$M(t), \text{ or } M_X(t), := E[e^{tX}]. \qquad [\mathbf{1}]$$

(ii) If $X, Y$ are independent with MGFs, $X+Y$ has MGF

$$\begin{aligned} M_{X+Y}(t) &:= E[e^{t(X+Y)}] \\ &= E[e^{tX} \cdot e^{tY}] \quad \text{(property of exponentials)} \\ &= E[e^{tX}] \cdot E[e^{tY}] \quad (e^{tX}, e^{tY} \text{ are independent as } X, Y \text{ are + Multiplication Th.)} \\ &= M_X(t) \cdot M_Y(t) : \end{aligned}$$

*the MGF of an independent sum is the product of the MGFs.* **[6]**

(iii) $N(0,1)$ has MGF

$$\begin{aligned} M(t) &= \frac{1}{\sqrt{2\pi}} \int \exp\{tx - \frac{1}{2}x^2\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int \exp\{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2\} dx \quad \text{(completing the square)} \\ &= \exp\{\frac{1}{2}t^2\} \cdot \frac{1}{\sqrt{2\pi}} \int \exp\{-\frac{1}{2}u^2\} du \qquad (u := x - t) \\ &= \exp\{\frac{1}{2}t^2\} \qquad \text{(normal density).} \end{aligned}$$

**[6]**

If $X \sim N(0,1)$, $(X-\mu)/\sigma \sim N(0,1)$, so has MGF $E[e^{t(X-\mu)/\sigma}] = e^{\frac{1}{2}t^2}$. Replace $t$ by $\sigma t$ and multiply by $e^{\mu t}$:

$$E[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} :$$

$N(\mu, \sigma^2)$ has MGF $\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$. **[3]**

(iv) By (ii) and (iii): $X+Y$ has MGF

$$\exp\{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2\} \cdot \exp\{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2\} = \exp\{(\mu_1 + \mu_2)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)t^2\}.$$

So $X+Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$: $X+Y$ is normal, with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. **[6]**

(v) The characteristic function (CF) of $X$ is defined by $\phi(t)$, or $\phi_X(t) := E[e^{itX}]$ ($t$ real). So to pass from MGF to CF, formally replace $t$ by $it$. This is justified here by *analytic continuation* (the MGF is entire, so the CF is entire). All the above goes through – e.g., $N(\mu, \sigma^2)$ has CF $\exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}$. **[3]**

[Seen – lectures]

Q3. (i)

$$\begin{aligned}\psi(t) = E[e^{itY}] &= E[\exp\{it(X_1 + \ldots + X_N)\}] \\ &= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n] \cdot P(N = n) \\ &= \sum_n e^{-\lambda}\lambda^n/n! \cdot E[\exp\{it(X_1 + \ldots + X_n)\}] \\ &= \sum_n e^{-\lambda}\lambda^n/n! \cdot (E[\exp\{itX_1\}])^n \\ &= \sum_n e^{-\lambda}\lambda^n/n! \cdot \phi(t)^n \\ &= \exp\{-\lambda(1 - \phi(t))\}. \end{aligned}$$ **[10]**

(ii) Differentiate:

$$\psi'(t) = \psi(t) \cdot \lambda\phi'(t),$$

$$\psi''(t) = \psi'(t) \cdot \lambda\phi'(t) + \psi(t) \cdot \lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda \cdot i\mu,$$

and as also $\psi'(0) = iEY$, this gives

$$EY = \lambda\mu.$$ **[8]**

(iii) Similarly,

$$\psi''(0) = i\lambda\mu \cdot i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2].$$ **[7]**

*Aliter*. Given $N$, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As $N$ is Poisson with parameter $\lambda$, $N$ has mean $\lambda$ and variance $\lambda$. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN \cdot var\ X + var\ N \cdot (EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda \cdot (EX)^2 = \lambda E[X^2].$$

[Seen – Problems]

Q4. (i) The transition probabilities are given by

$$p_{i,i-1} = \frac{i}{d}, \qquad p_{i,i+1} = \frac{d-i}{d}, \qquad p_{ij} = 0 \quad \text{otherwise.} \qquad \textbf{[3]}$$

(ii) There is no limit distribution, as the chain is periodic with period 2.

$$\binom{d}{j-1}(d-j+1) = \frac{d!}{(j-1)!(d-j+1)!}.(d-j+1) = \frac{d!}{(j-1)!(d-j)!} = \binom{d}{j}.j, \qquad (a)$$

$$\binom{d}{j+1}(j+1) = \frac{d!}{(j+1)!(d-j-1)!}(j+1) = \frac{d!}{j!(d-j-1)!} = \binom{d}{j}.(d-j). \qquad (b)$$

So by (a) and (b) ($\pi$ is a prob. distribution, by the binomial theorem),

$$(\pi P)_j = \sum_i \pi_i p_{ij} = \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j}$$

$$= 2^{-d}\binom{d}{j-1}\frac{(d-j+1)}{d} + 2^{-d}\binom{d}{j+1}\frac{(j+1)}{d} = \frac{2^{-d}}{d}\binom{d}{j}\{j+(d-j)\} = 2^{-d}\binom{d}{j}$$

$$= \pi_j.$$

This says that $\pi P = \pi$, so $\pi$ is invariant. **[7]**

(iii)

$$\pi_i p_{i,i+1} = 2^{-d}\binom{d}{i}.\frac{d-i}{d} = 2^{-d}\frac{d!}{(d-i)!i!}.\frac{d-i}{d} = 2^{-d}\binom{d-1}{i},$$

$$\pi_{i+1} p_{i+1,i}.\frac{i+1}{d} = 2^{-d}\binom{d}{i+1} = 2^{-d}\frac{(d-1)!}{i!(d-i-1)!}.\frac{i+1}{d} = 2^{-d}\binom{d-1}{i},$$

proving detailed balance, and so reversibility. **[7]**

Hence we can calculate the invariant distribution (unique by $(DB)$):

$$i = 0: \quad \pi_1 = \pi_0 \frac{p_{01}}{p_{10}} = \frac{\pi_0}{\frac{1}{d}}; \qquad i = 1: \quad \pi_2 = \pi_1 \frac{p_{12}}{p_{21}} = \frac{\pi_0}{\frac{1}{d}}.\frac{1-\frac{1}{d}}{\frac{2}{d}}, \ldots,$$

$$\pi_i = \frac{\pi_0}{\frac{1}{d}}.\frac{1-\frac{1}{d}}{\frac{2}{d}}.\ldots.\frac{1-\frac{i-1}{d}}{\frac{i}{d}} = \pi_0.\frac{d(d-1)\ldots(d-i+1)}{1.2.\ldots.i} = \pi_0\binom{d}{i}.$$

Then $\sum_i \pi_i = 1$ gives

$$\pi_0 \sum_i \binom{d}{i} = \pi_0.2^d = 1, \qquad \pi_0 = 2^{-d}, \qquad \pi_i = 2^{-d}\binom{d}{i}. \qquad \textbf{[8]}$$

[Seen – Problems]