pfsl21.tex

Lecture 21. 27.11.2013

The proof is not examinable, and is on the handout (cf. [BK], 5.3.1; SP L20-22). It gives the Paley-Wiener-Zygmund (PWZ) construction of 1933, and is a streamlined version of the classical one due to Lévy in his book of 1948 and Cieselski in 1961. It formalises in the modern language of *wavelets* Lévy's *broken-line* construction.

2. Poisson process; compound Poisson processes

Exponential Distribution

A random variable T is said to have an exponential distribution with rate λ, or $T \sim E(\lambda)$, if

$$P(T \leq t) = 1 - e^{-\lambda t} \qquad \text{for all } \; t \geq 0.$$

Recall $E(T) = 1/\lambda$ and $var(T) = 1/\lambda^2$. Further important properties are:
(i) Exponentially distributed random variables possess the 'lack of memory' property: $P(T > s + t | T > t) = P(T > s)$ (below).
(ii) Let $T_1, T_2, \ldots T_n$ be independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ resp. Then $\min\{T_1, T_2, \ldots, T_n\}$ is exponentially distributed with rate $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.
(iii) Let $T_1, T_2, \ldots T_n$ be independent exponentially distributed random variables with parameter λ. Then $G_n = T_1 + T_2 + \ldots + T_n$ has a $Gamma(n, \lambda)$ distribution. That is, its density is

$$P(G_n = t) = \lambda e^{-\lambda t} (\lambda t)^{n-1}/(n-1)! \qquad \text{for } \; t \geq 0.$$

The Poisson Process

Definition. Let $t_1, t_2, \ldots t_n$ be independent exponential $E(\lambda)$ random variables, $T_n := t_1, + \ldots + t_n$ for $n \geq 1$, $T_0 = 0$, $N(s) := \max\{n : T_n \leq s\}$.
Interpretation: Think of t_i as the time between arrivals of events, then T_n is the arrival time of the nth event and $N(s)$ the number of arrivals by time s. Then $N(s)$ has a Poisson distribution with mean λs. The Poisson process can also be characterised via

Theorem. If $\{N(s), s \geq 0\}$ is a Poisson process, then
(i) $N(0) = 0$,
(ii) $N(t + s) - N(s)$ is Poisson $P(\lambda t)$, and
(iii) $N(t)$ has independent increments.
Conversely, if (i),(ii) and (iii) hold, then $\{N(s), s \geq 0\}$ is a Poisson process.

The above characterization can be used to extend the definition of the Poisson process to include time-dependent intensities. We say that $\{N(s), s \geq 0\}$ is a *Poisson process* with *rate* $\lambda(r)$ if
(i) $N(0) = 0$,
(ii) $N(t+s) - N(s)$ is Poisson with mean $\int_s^t \lambda(r)dr$, and
(iii) $N(t)$ has independent increments.

Compound Poisson Processes

We now associate i.i.d. random variables Y_i with each arrival and consider

$$S(t) = Y_1 + \ldots + Y_{N(t)}, \qquad S(t) = 0 \text{ if } N(t) = 0.$$

Theorem. Let (Y_i) be i.i.d. and N be an independent nonnegative integer random variable, and S as above.
(i) If $E(N) < \infty$, then $E(S) = EX(N).E(Y_1)$.
(ii) If $E(N^2) < \infty$, then $var(S) = E(N).var(Y_1) + var(N)(E(Y_1))^2$.
(iii) If $N = N(t)$ is Poisson(λt), then $var(S) = t\lambda(E(Y_1))^2$.

A typical application in the insurance context is a Poisson model of claim arrival with random claim sizes.

Renewal Processes

Suppose we use components – light-bulbs, say – whose lifetimes $X_1, X_2, \ldots$ are independent, all with law F on $(0, \infty$. The first component is installed new, used until failure, then replaced, and we continue in this way. Write

$$S_n := \sum_1^n X_i, \qquad N_t := \max\{k : S_k < t\}.$$

Then $N = (N_t : t \geq 0)$ is called the *renewal process* generated by F; it is a *counting process*, counting the number of failures seen by time t.

The law F has the *lack-of-memory property* iff the components show no aging – that is, if a component still in use behaves as if new. The condition for this is

$$P(X > s + t | X > s) = P(X > t) \qquad (s, t > 0),$$

or

$$P(X > s + t) = P(X > s)P(X > t).$$

Writing $\overline{F}(x) := 1 - F(x)$ $(x \geq 0)$ for the *tail* of F, this says that

$$\overline{F}(s + t) = \overline{F}(s)\overline{F}(t) \qquad (s, t \geq 0).$$

Obvious solutions are

$$\overline{F}(t) = e^{-\lambda t}, \qquad F(t) = 1 - e^{-\lambda t}$$

for some $\lambda > 0$ – the exponential law $E(\lambda)$. Now

$$f(s+t) = f(s)f(t) \qquad (s, t \geq 0)$$

is a 'functional equation' – the *Cauchy functional equation* – and we quote that these are the *only* solutions, subject to minimal regularity (such as one-sided boundedness, as here – even on an interval of arbitrarily small length!).

So the exponential laws $E(\lambda)$ are *characterized* by the lack-of-memory property. Also, the lack-of-memory property corresponds in the renewal context to the *Markov property.* The renewal process generated by $E(\lambda)$ is called the *Poisson (point) process* with *rate* λ, $Ppp(\lambda)$. So: among renewal processes, the only Markov processes are the Poisson processes. We meet Lévy processes below: among renewal processes, the only Lévy processes are the Poisson processes.

It is the lack of memory property of the exponential distribution that (since the inter-arrival times of the Poisson process are exponentially distributed) makes the Poisson process the basic model for events occurring 'out of the blue'. Typical examples are accidents, insurance claims, hospital admissions, earthquakes, volcanic eruptions etc. So it is not surprising that Poisson processes and their extensions (compound Poisson processes) dominate the theory of the actuarial and insurance professions, as well as geophysics, etc.

3. Lévy Processes; Lévy-Itô decomposition

Lévy Processes

Suppose we have a process $X = (X_t : t \geq 0)$ that has stationary independent increments. Such a process is called a *Lévy process*, in honour of their creator, the great French probabilist Paul Lévy. Then for each $n = 1, 2, \ldots,$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \ldots + (X_t - X_{(n-1)t/n})$$

displays X_t as the sum of n independent (by independent increments), identically distributed (by stationary increments) random variables. Consequently, X_t is *infinitely divisible*, so its CF is given by the Lévy-Khintchine formula. Prime example: the Wiener process [= Brownian motion] is a Lévy process.

Poisson Processes.

The increment $N_{t+u} - N_u$ $(t, u \geq 0)$ of a Poisson process is the number of failures in $(u, t+u]$ (in the language of renewal theory). By the lack-of-memory property of the exponential, this is independent of the failures in $[0, u]$, so the increments of N are *independent.* It is also identically distributed to the number of failures in $[0, t]$, so the increments of N are *stationary.* That is, N has stationary independent increments, so is a Lévy process: Poisson processes are Lévy processes.

We need an important property: two Poisson processes (on the same filtration) are independent iff they never jump together (a.s.).

The Poisson count in an interval of length t is Poisson $P(\lambda t)$ (where the rate λ is the parameter in the exponential $E(\lambda)$ of the renewal-theory viewpoint), and the Poisson counts of disjoint intervals are independent. This extends from intervals to Borel sets:
(i) For B Borel, the Poisson count in B is Poisson $P(\lambda|B|)$, with $|.|$ Lebesgue measure; (ii) Poisson counts over disjoint Borel sets are independent.

Poisson (Random) Measures.

If ν is a finite measure, call a random measure ϕ *Poisson* with *intensity* (or characteristic) *measure* ν if for each Borel set B, $\phi(B)$ has a Poisson distribution with parameter $\nu(B)$, and for $B_1, \ldots, B_n$ disjoint, $\phi(B_1), \ldots, \phi(B_n)$ are independent. One can extend to σ-finite measures ν: if (E_n) are disjoint with union $\mathbf{R}$ and each $\nu(E_n) < \infty$, construct ϕ_n from ν restricted to E_n and write ϕ for $\sum \phi_n$.

Poisson Point Processes.

With ν as above a (σ-finite) measure on $\mathbf{R}$, consider the product measure $\mu = \nu \times dt$ on $\mathbf{R} \times [0, \infty)$, and a Poisson measure ϕ on it with intensity μ. Then ϕ has the form

$$\phi = \sum_{t \geq 0} \delta_{(e(t),t)},$$

where the sum is *countable.* Thus ϕ is the sum of Dirac measures over 'Poisson points' $e(t)$ occurring at Poisson times t. Call $e = (e(t) : t \geq 0)$ a *Poisson point process* with *characteristic measure* ν,

$$e = Ppp(\nu).$$

For each Borel set B, define the *counting process* of B:

$$N(t, B) := \phi(B \times [0, t]) = card\{s \leq t : e(s) \in B\}.$$