## pfsl22.tex Lecture 22. 28.11.2013

This counts the Poisson points in B – and is a Poisson process with rate (parameter)  $\nu(B)$ . All this reverses: starting with an  $e = (e(t) : t \ge 0)$  whose counting processes over Borel sets B are Poisson  $P(\nu(B))$ , then – as no point can contribute to more than one count over disjoint sets – disjoint counting processes never jump together, so are independent by above, and  $\phi := \sum_{t>0} \delta_{(e(t),t)}$  is a Poisson measure with intensity  $\mu = \nu \times dt$ .

## Lévy Processes; Lévy-Khintchine Formula; Lévy-Itô decomposition.

We can now sketch the close link between the general Lévy process on the one hand and the general infinitely-divisible law given by the Lévy-Khintchine formula (LK) on the other.

First, if  $X = (X_t)$  is Lévy, the law of each  $X_1$  is infinitely divisible, so

$$E\exp\{iuX_1\} = \exp\{-\Psi(u)\} \qquad (u \in \mathbf{R})$$

with  $\Psi$  a Lévy exponent as in (LK). Similarly,

$$E\exp\{iuX_t\} = \exp\{-t\Psi(u)\} \qquad (u \in \mathbf{R}),$$

for rational t at first and general t by approximation and càdlàg paths. Then  $\Psi$  is called the *Lévy exponent*, or *characteristic exponent*, of the Lévy process X. Conversely, given a Lévy exponent  $\Psi(u)$  as in (LK), construct a Brownian motion, and an independent Ppp  $\Delta = (\Delta_t : t \ge 0)$  with characteristic measure  $\mu$ , the Lévy measure in (LK). Then  $X_1(t) := at + \sigma B_t$  has CF

$$E \exp\{iuX_1(t)\} = \exp\{-t\Psi_1(t)\} = \exp\{-t(iau + \frac{1}{2}\sigma^2 u^2)\},\$$

giving the non-integral terms in (LK). For the 'large' jumps of  $\Delta$ , write

$$\Delta_t^{(2)} := \Delta_t \text{ if } |\Delta_t| \ge 1, \quad 0 \text{ else.}$$

Then  $\Delta^{(2)}$  is a Poisson point process with characteristic measure  $\mu^{(2)}(dx) := I(|x| \ge 1)\mu(dx)$ . Since  $\int \min(1, |x|^2)\mu(dx) < \infty, \mu^{(2)}$  has finite mass, so  $\Delta^{(2)}$ , a  $Ppp(\mu^{(2)})$ , is discrete and its counting process

$$X_t^{(2)} := \sum_{s \le t} \Delta_s^{(2)} \qquad (t \ge 0)$$

is compound Poisson, with Lévy exponent

$$\Psi^{(2)}(u) = \int (1 - e^{iux}) I(|x| \ge 1) \mu(dx) = \int (1 - e^{iux}) \mu^{(2)}(dx).$$

There remain the 'small jumps',

$$\Delta_t^{(3)} := \Delta_t \quad \text{if} \quad |\Delta_t| < 1, \quad 0 \text{ else},$$

a  $Ppp(\mu^{(3)})$ , where  $\mu^{(3)}(dx) = I(|x| < 1)\mu(dx)$ , and independent of  $\Delta^{(2)}$  because  $\Delta^{(2)}$ ,  $\Delta^{(3)}$  are Poisson point processes that never jump together. For each  $\epsilon > 0$ , the 'compensated sum of jumps'

$$X_t^{(\epsilon,3)} := \sum_{s \le t} I(\epsilon < |\Delta_s| < 1)\Delta_s - t \int x I(\epsilon < |x| < 1)\mu(dx) \qquad (t \ge 0)$$

is a Lévy process with Lévy exponent

$$\Psi^{(\epsilon,3)}(u) = \int (1 - e^{iux} + iux)I(\epsilon < |x| < 1)\mu(dx).$$

Use of a suitable maximal inequality allows passage to the limit  $\epsilon \downarrow 0$  (going from finite to possibly countably infinite sums of jumps):  $X_t^{(\epsilon,3)} \to X_t^{(3)}$ , a Lévy process with Lévy exponent

$$\Psi^{(3)}(u) = \int (1 - e^{iux} + iux)I(|x| < 1)\mu(dx),$$

independent of  $X^{(2)}$  and with càdlàg paths. Combining:

Theorem (Lévy-Itô decomposition). For a Lévy exponent

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI(|x| < 1)\mu(dx)),$$

the construction above yields a Lévy process

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

with Lévy exponent  $\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$ . Here the  $X^{(i)}$  are independent Lévy processes, with Lévy exponents  $\Psi^{(i)}$ ;  $X^{(1)}$  is Gaussian,  $X^{(2)}$  is a compound Poisson process with jumps of modulus  $\geq 1$ ;  $X^{(3)}$  is a compensated sum of jumps of modulus < 1. The jump process  $\Delta X = (\Delta X_t : t \ge 0)$  is a  $Ppp(\mu)$ , and similarly  $\Delta X^{(i)}$  is a  $Ppp(\mu^{(i)})$  for i = 2, 3.

Stable processes.

A stable process has (to within location and scale) a Lévy exponent involving two parameters,  $\alpha \in (0, 2]$ , called the *index*, and  $\beta \in [-1, 1]$ , called the *skewness parameter*:

$$\Psi(u) = |u|^{\alpha} (1 - i\beta(sgn u) \tan(\frac{1}{2}\pi\alpha)) \quad (\alpha \neq 1), \quad |u|(1 + i\beta(sgn u)\frac{2}{\pi}\log|u|) \quad (\alpha = 1)$$

(for  $\alpha = 2$ ,  $\beta$  drops out as tan  $\pi = 0$ , so  $\Psi(u) = u^2$ , giving the normal (Gaussian) distribution). The case  $\alpha = 1$  is the *Cauchy* case; the asymmetric Cauchy case  $\alpha = 1, \beta \neq 0$  is awkward, and we do not consider it further.

The Lévy measure  $\mu$  in the stable case is absolutely continuous, with density  $\nu$ ,  $\mu(dx) = \nu(x)dx$ , where

$$\nu(x) = c_+/x^{1+\alpha}$$
 (x > 0),  $c_-/|x|^{1+\alpha}$  (x < 0) ( $c_{\pm} \ge 0, c_+ + c_- > 0$ ).

Here

$$\beta = (c_{=} - c_{-})/(c_{+} + c_{-}).$$

The calculations are simpler in the symmetric case,  $c_{+} = c_{-} = c$  say. Then

$$\Psi'(u) = 2cu^{\alpha-1}I \qquad (u>0), \qquad I := \int_0^\infty v^{-\alpha}\sin v dv$$

So  $\Psi(u) = 2cIu^{\alpha}/\alpha$  for u > 0, and similarly for u < 0:  $\Psi(u) = |u|^{\alpha}.2cI/a$ But (see e.g. M2PM3 L30 on my website: there  $t = 1 - \alpha \in (0, 1)$ , but we can extend by analytic continuation to -1 < t < 1,  $\alpha \in (0, 2)$ )  $I = \Gamma(1-\alpha)\cos(\frac{1}{2}\pi\alpha)$  (here  $\alpha \neq 1$ :  $\Gamma(z)$  has a pole at z = 0; for the Cauchy case  $\alpha = 1$  see above). Choose  $c := \sigma/(2I)$ ; then  $\Psi(u) = |u|^{\alpha}$ .

Example. The symmetric stable law with  $\alpha = 3/2$  is called the *Holtsmark distribution*, proposed by the Danish physicist J. Holtsmark in 1919 as a model for the distribution of galaxies in space (here 3/2 comes from the 3 dimensions of space and the 2 in Newton's Inverse Square Law of Gravity). Since  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  and  $\Gamma(1 + x) = x\Gamma(x)$ ,  $\Gamma(1 - \alpha)\cos(\frac{1}{2}\pi\alpha)$  here is  $\int_0^\infty v^{3/2}\sin v dv = \Gamma(-\frac{1}{2})\cos(3\pi/4) = (-2\sqrt{\pi}).(-1/\sqrt{2}) = \sqrt{2\pi}$ . Subordinators.

We resort to complex numbers in the CF  $\phi(u) = E(e^{iuX})$  because this always exists – for all real u – unlike the ostensibly simpler moment-generating

function (MGF)  $M(u) := E(e^{uX})$ , which may well diverge for some real u. However, if the random variable X is *non-negative*, then for  $s \ge 0$  the Laplace-Stieltjes transform (LST)

$$\psi(s) := E[e^{-sX}] \le E(1) = 1$$

always exists. For  $X \ge 0$  we have both the CF and the LST to hand, but the LST is usually simpler to handle. We can pass from CF to LST formally by taking u = is, and this can be justified by analytic continuation.

Some Lévy processes X have increasing (i.e. non-decreasing) sample paths; these are called *subordinators*. From the construction above, subordinators can have no negative jumps, so  $\mu$  has support in  $(0, \infty)$  and no mass on  $(-\infty, 0)$ . Because increasing functions have FV, one must have paths of (locally) finite variation, the condition for which can be shown to be

$$\int \min(1,|x|)\mu(dx) < \infty.$$

Thus the Lévy exponent must be of the form

$$\Psi(u) = -idu + \int_0^\infty (1 - e^{iux})\mu(dx),$$

with  $d \ge 0$ . It is more convenient to use the Laplace exponent  $\Phi(s) = \Psi(is)$ :

$$E(\exp\{-sX_t\}) = \exp\{-t\Phi(s)\} \qquad (s \ge 0), \qquad \Phi(s) = ds + \int_0^\infty (1 - e^{-sx})\mu(dx)$$

Example. The Stable Subordinator.

Here  $d = 0, \Phi(s) = s^{\alpha}, \ (0 < \alpha < 1),$ 

$$\mu(dx) = dx / (\Gamma(1 - \alpha)x^{\alpha - 1})$$

(in one normalization, the one convenient here). The special case  $\alpha = 1/2$  is particularly important: this arises as the first-passage time of Brownian motion over positive levels, and gives rise to the *Lévy density* (see Problems). *Classification.* 

*IV (Infinite Variation).* The sample paths have infinite variation on finite time-intervals, a.s. This occurs iff

$$\sigma > 0$$
 or  $\int \min(1, |x|)\mu(dx) = \infty$ .