pfsl25.tex

**Lecture 25. 5.12.2013**

*Ehrenfest urn (continued).* Think of the urns as two equal parts of a symmetrical container, $A$ and $B$. If $A$ has an excess (more than a half) of molecules, it is more likely to lose molecules to $B$ than gain them, and similarly for $B$. The model exhibits a *central force*, or *restoring force*, towards equilibrium (equal numbers in the two halves), but allows departures from equilibrium by spontaneous fluctuations. We return to the physical interpretation, which is very important, later.

6. *Bernoulli-Laplace model of diffusion* (Daniel BERNOULLI (1700-1782) in 1769, Laplace in 1812). This is a more complicated version of the Ehrenfest model, though it came much earlier. Here there are $2N$ balls of two colours – $N$ white balls and $N$ black balls, say, and two containers, each containing $N$ balls. At each stage, a ball is chosen at random from each container, and they are interchanged. The state is the number of white balls (say) in the first container (say):

$$p_{i,i-1} = (i/N)^2, \quad p_{i,i} = 2N(N-i)/N^2, \quad p_{i,i+1} = (N-i)^2/N^2, \quad p_{i,j} = 0 \quad \text{otherwise.}$$

Again, the matrix is tri-diagonal.

7. *Wright-Fisher model in mathematical genetics.* This was introduced by the American geneticist Sewall G. WRIGHT (1889-1988) in 1931, and the English statistician and geneticist R. A. (Sir Ronald) Fisher (1890-1962) in 1930. For more on mathematical genetics, see the handout.

8. *Motor insurance.* Suppose that a motorist has probability $p$ of driving for a year without making a motor insurance claim ('success'), and probability $q := 1 - p$ of claiming ('failure'). For each claim-free year, he is rewarded by a reduction in his premium (until he reaches some minimum premium); for each claim, he is penalized by going back to the starting premium. One may model this with a transition matrix with $p$ in the super-diagonal and $q$ in the first column.

9. *Social mobility.* Markov chains are used by sociologists to study social mobility between classes.

*Non-Markovian phenomena.* These are typically much harder, as the past history of the process is now relevant as well as its present position. One important example is familiar from physics: *hysteresis.* With a hysteresis loop, one needs to know not just what level one is at, but whether one is going 'up or down'. Sometimes (as here), one can recover the Markov property by including such extra information – but at the price of working with a more complicated state-space.

**2. Classification of states**.

We say state $E_k$ (which we may as well abbreviate to $k$ from now on) can be *reached* from $j$ – or, is *accessible* from $j$, or $j$ *leads to* $k$ – if $p_{jk}^{(n)} > 0$ – i.e. there is positive probability of a transition from $j$ to $k$ in $n$ steps for some $n$. We then write $j \to k$. If also $k \to j$, we say that $j$ and $k$ *inter-communicate*, and write $j \leftrightarrow k$.

Call a set $C$ of states *closed* if no states outside $C$ are accessible from any state in $C$ – that is, once the process enters $C$ it stays there. The *closure* $\bar{C}$ of any set $C$ of states is the smallest closed set containing it.

A singleton closed set is an *absorbing state*, or *trap* (example: the extreme states 0 and $a$ in the gambler's ruin problem).

A Markov chain is called *irreducible* if there is no closed set other than the set of all states (the motivation for this term will become clearer later when we have proved the Classification Theorem).

*Subchains.*

Recall that a matrix $Q$ is *stochastic* if its elements are non-negative and its row-sums are 1 (example: the transition probability matrix $P = (p_{ij})$ of a Markov chain)).

If we have a closed set $C$ of states, let us for convenience label states in $C$ first, then states outside $C$. Then the transition matrix $P$ has the form

$$P = \begin{pmatrix} Q & 0 \\ U & V \end{pmatrix},$$

for some $U$ and $V$. Here $Q$ governs transitions from $C$ to $C$, the 0 reflects the impossibility of leaving $C$, $U$ governs transitions from outside $C$ to $C$, $V$ from outside $C$ to outside $C$. Note that $P^n$ has the form

$$P^n = \begin{pmatrix} Q^n & 0 \\ U^{(n)} & V^{(n)} \end{pmatrix},$$

where $P^n$, $Q^n$ are matrix powers but $U^{(n)}$, $V^{(n)}$ are not in general.

We can now imagine *deleting* all states outside $C$ from the state space. We are left with a Markov chain, with state space $C$ and transition matrix $Q$. It is called the Markov chain *restricted* to $C$, or the *sub-chain* on $C$.

*Periodicity.*

A state $j$ has *period* $t > 1$ if $p_{jj}^{(n)} = 0$ unless $n$ is a multiple of $t$, and $t$ is the largest such integer. Otherwise, we call $j$ *aperiodic* ('period $t = 1$').

*Examples.* 1. In simple random walk, all states are periodic with period 2. Similarly for the Ehrenfest model of diffusion.
2. If we modify our random walk to have positive probabilities of moving left, right and staying put, all states are now aperiodic.
3. For

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

we find by direct calculation

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

So all the elements of $P^n$ for $n \geq 2$ are positive, and so all states are aperiodic – even though $P$ has zeros on its diagonal.
*Persistence and transience.*
Call $j$ *persistent* (or *recurrent*) if

$$P(\text{return to } j | \text{start at } j) = 1$$

– return to $j$ from $j$ is certain. Then return to $j$ $n$ times is certain for each $n$, and

$$P(\text{return to } j \ i.o. | \text{start at } j) = 1.$$

Call $j$ *transient* if

$$f_j := P(\text{return to } j | \text{start at } j) < 1.$$

Then

$$P(\text{return to } j \ n \text{ times} | \text{start at } j) = f_j^n \to 0 \qquad (n \to \infty),$$

and

$$P(\text{return to } j \text{ i.o} | \text{start at } j) = 0.$$

Writing $T_j$ for the first return time to $j$, state $j$ is persistent iff $T_j$ is non-defective, transient iff $T_j$ is defective. Just as for random walks, if

$$u_{n,j} := P(\text{at } j \text{ at time } n \mid \text{start at } j), \qquad U_j(s) := \sum\nolimits_{n=0}^{\infty} u_{n,j} s^n,$$

then for

$$F_j(s) := E[s^{t_j}] = \sum\nolimits_0^\infty P(T_j = n)s^n = \sum\nolimits_0^\infty f_{n,j}s^n,$$

we have the Feller relation (Problems 9 Q1)

$$U_j(s) = 1/(1 - F_j(s)).$$

So $U_j(1) < \infty$ iff $F_j(1) < 1$, iff $T_j$ is defective, iff $j$ is transient – that is, $\sum_n u_{n,j}$ converges iff $j$ is transient, so diverges iff $j$ is persistent.

*The Erdös-Feller-Pollard Theorem.*

If $j$ is periodic with period $t$, $u_n = 0$ if $n$ is not a multiple of $t$. Write

$$\mu_j := E[T_j] = F_j'(1)$$

for the mean return time to $j$ (or mean recurrence time of $j$). We call state $j$ *null* if $\mu_j = \infty$, *positive* if $\mu_j < \infty$.

*Examples*: In symmetric random walk, all states are null recurrent.

This terminology of null and positive is explained by the next result, proved by Paul ERDÖS (Hungarian mathematician, 1913-1996), Willy FELLER (Yugoslav/US mathematician, 1906-1970) and Harry POLLARD (American mathematician, d. 1985) in 1949.

**Theorem (Erdös-Feller-Pollard Theorem)**. If state $j$ is persistent:

(i) If $j$ is aperiodic,

$$u_{n,j} \to 1/\mu_j \qquad (n \to \infty).$$

(ii) If $j$ is periodic with period $t$, $u_{n,j} = 0$ unless $n$ is a multiple of $t$, and

$$u_{nt,j} \to t/\mu_j \qquad (n \to \infty).$$

*Sketch Proof.* (i) By the Feller relation,

$$(1 - s)U_j(s) = (1 - s)/(1 - F_j(s)).$$

As $s \uparrow 1$, the RHS tends to $1/F_j'(1) = 1/\mu_j$. The left is (recall $u_{0,j} = 0$)

$$(1 - s)U_j(s) = \sum\nolimits_n u_{n,j}s^n - \sum\nolimits_n u_{n,j}s^{n+1} = \sum\nolimits_n (u_{n,j} - u_{n-1,j})s^n.$$

As $s \uparrow 1$, this tends to

$$\sum\nolimits_n (u_{n,j} - u_{n-1,j}) = \lim_{N\to\infty}\sum\nolimits_{n=0}^N (u_{n,j} - u_{n-1,j}) = \lim_{N\to\infty} u_{n,j}.$$

This establishes the result formally in the aperiodic case. A full proof needs more care. For details (not examinable), see e.g. [GS] 5.2, 5.10 or [F] XIII.11. The periodic case is similar. //