pfsl26.tex

Lecture 26. 5.12.2013 (half-hour: Problems)

This gives us an easy way, in the persistent case, to tell the two sub-cases of null and positive apart. If j is null,

$$u_{n,j} \to 0,$$

while if j is positive,

$$u_{n,j} \to 1/\mu_j > 0$$

(in the aperiodic case, with a similar statement in the periodic case). This will be useful below. It also explains the terms *null* and *positive*.

We introduce one more term (the motivation is from Physics, specifically Statistical Mechanics, to which we return later). A state is called *ergodic* if it is aperiodic and positive recurrent (= persistent).

When a chain is irreducible (so each state can be reached from every other state, eventually), we quote that all states have the same character: all aperiodic/periodic with the same period, all transient, all recurrent, all null, all positive, or all ergodic. Results of this type are called *solidarity theorems*; we shall assume them. We then call an irreducible chain aperiodic etc. if all its states are.

3. Limit distributions and invariant (= stationary) distributions

Recall that the transition probability matrix P of a Markov chain has row-sums 1. This means that if we post-multiply P by the column-vector $\mathbf{1}$ all of whose elements are 1,

$$P\mathbf{1} = \mathbf{1}.$$

This says that 1 is an *eigenvalue*, with right *eigenvector* $\mathbf{1}$.

It turns out that this eigenvalue is special, and that the long-term behaviour of the chain is dominated by the eigenstructure of P. The key result is the following classical theorem, which (perhaps surprisingly) is a result of Linear Algebra. It is due to Oskar PERRON (1880-1975) in 1907 and Georg FROBENIUS (1849-1917) in 1908 and 1912.

Theorem (Perron-Frobenius Theorem). Let P be the transition probability matrix of a finite irreducible Markov chain with period d.
(i) $\lambda_1 = 1$ is always an eigenvalue of P; if $d > 1$, so too are the other dth roots of unity, $\lambda_2 = \omega, \ldots, \lambda_d = \omega^{d-1}$, where $\omega := \exp\{2\pi i/d\}$.
(ii) All other eigenvalues λ_j have modulus $|\lambda_j| < 1$.

The eigenvalue (e-value) 1 is called the *Perron-Frobenius (PF)* e-value. For proof of the PF theorem, see e.g. [HJ], 8.4, [Sen].

Theorem. In an ergodic chain (not necessarily finite):
(i) there exists

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j,$$

independent of i.
(ii) $\pi_j > 0$, and $\sum \pi_j = 1$.
(iii) $\pi_j = \sum_i \pi_i p_{ij}$ for each j, or writing π for the row-vector of the π_j,

$$\pi = \pi P.$$

Thus π is the *left* eigenvector for the Perron-Frobenius eigenvalue 1.

Conversely, if (ii), (iii) hold for an irreducible periodic chain, (i) holds, with $\pi_k = 1/\mu_k$, μ_k the mean recurrence time of state k, and the chain is ergodic.

Proof. $P_{ij}(s) = F_{ij}(s)P_{jj}(s)$. By the Erdös-Feller-Pollard theorem,

$$p_{jj}^{(n)} \to \pi_j = 1/\mu_j \qquad (n \to \infty),$$

and $\pi_j > 0$ as $\mu_j < \infty$ (the states are positive, as the chain is ergodic). So if

$$f_{ij} = F_{ij}(1) = P_i(\text{reach } j),$$

$$p_{ij}^{(n)} \sim f_{ij} p_{jj}^{(n)} \to f_{ij}\pi_j.$$

But here $f_{ij} = 1$ as the chain is irreducible (each state is accessible from every other), so

$$p_{ij}^{(n)} \to \pi_j \qquad (n \to \infty)$$

for each i, proving (i). Now

$$1 = \sum_{j=1}^{\infty} p_{ij}^{(n)} \geq \sum_{j=1}^{N} p_{ij}^{(n)}$$

for each N. Let $n \to \infty$:

$$1 \geq \sum_{j=1}^{N} \pi_j,$$

for each N.