pfsl4.tex

Lecture 4. 16.10.2013

Higher dimensions; joint and marginal distributions

If $X = (X_1, \ldots, X_n)$ is a random variable taking values in n-dimensional space – a random n-vector – then its distribution function F is defined as above, but coordinatewise. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we write

$$x \leq y \text{ iff } x_1 \leq y_1, \ldots, x_n \leq y_n.$$

Then

$$F(x) := P(X \leq x) = P(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

This is also called the *joint* distribution of $(X_1, \ldots, X_n)$, while

$$F_i(x_i) := P(X_i \leq x_i), \qquad i = 1, \ldots, n$$

is called the *marginal* distribution of X_i. Note that letting the jth argument $x_j \to \infty$ eliminates the condition $X_j \leq x_j$, and so leaves the joint distribution of the Xs with X_j omitted. So the joint distribution of a random vector determines the joint distribution of any subvector, and the marginals of its coordinates, just by letting unwanted arguments go to $+\infty$. In sum: *the joint determines the marginals.*

Probability Integral Transformation (PIT).

As F is non-decreasing, it has an inverse function. We use

$$F^{-1}(x) := \inf\{x : F(x) \geq t\} = \min\{x : F(x) \geq t\}$$

(also non-decreasing, but left-continuous – so the infinum is attained, i.e. is a minimum). Write $X \sim F$ to mean that the random variable X has distribution F. Then if $U[0, 1]$ is the uniform distribution above (probability = length) and $U \sim U[0, 1]$, then U is uniformly distributed on $[0, 1]$; we shall use this standard notation below. The *Probability Integral Transformation (PIT)* uses U and F to generate X:

$$X := F^{-1}(U) \sim F. \qquad (PIT).$$

Proof.

$$P(X = F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x). \qquad //$$

The PIT is very useful in the context of Simulation (using computers to generate random numbers); see IS, I and p.2. It means that we only need random number tables for the uniform distribution $U[0,1]$, and can then use (PIT) to transform this data to have distribution F.

Copulas

The question arises of how to go in the reverse direction. It is helpful to think of the information in the joint distribution as composed of two parts: one on the marginals, the other on the *dependence* between the coordinates – often of great statistical importance! One needs a function that *couples* the marginals together to form the joint. This is called the *copula*.

A *copula* C in n dimensions is a probability distribution function on (= supported by – all its probability mass is on) the unit n-cube $[0,1]^n$.

Sklar's Theorem (A. SKLAR, 1958). If $F(x)$ is a joint distribution in n dimensions, with marginals $F_i(x_i)$, there exists an n-dimensional copula C with

$$F(x) = F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Conversely, given any copula C and marginals F_i, this formula gives a joint distribution F with marginals F_i. The correspondence between F and C is unique if the marginals F_i are continuous.

Absolute continuity and the Radon-Nikodym theorem

In the density case,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du.$$

In the discrete case,

$$F(x) = P(X \leq x) = \sum_{n: x_n \leq x} f(x_n).$$

Each expresses a relationship between measures. In the density case, the measures are F and λ, Lebesgue measure:

$$\lambda(B) = 0 \quad \Rightarrow \quad F(B) = 0.$$

In the discrete case, the measures are F and counting measure on the set of values $\{n : x_n\}$ (think of $x_n = n$, say). In general: if P, Q are measures, we say Q is *absolutely continuous* w.r.t. P, written $Q << P$, if

$$P(A) = 0 \quad \Rightarrow Q(A) = 0.$$

Then the *Radon-Nikodym theorem* states that $Q << P$ iff

$$Q(A) = \int_A f dP$$

for some measurable function f, called the *Radon-Nikodym (RN) derivative* of Q w.r.t. P, written $f = dQ/dP$. Thus each of the fs above is a RN derivative. See e.g. SP L7, or [S] Ch. 19.

II. DISTRIBUTIONS AND THEIR TRANSFORMS

1. Examples.

1. *Uniform* $U[a, b]$. This has density

$$f(x) = 1/(b - a) \qquad (a \le x \le b), \qquad 0 \qquad \text{otherwise}$$

and distribution

$$F(x) = 0 \qquad (x \le a), \qquad (x-a)/(b-a) \qquad (a \le x \le b), \qquad 1 \qquad (x \ge b).$$

The case $U[0, 1]$ is basic – we have met this in I, and seen how to get any other distribution from it by the Probability Integral Transformation.

$U[a, b]$ forms a two-parameter family. It is statistically interesting, as maximum-likelihood estimation (MLE) of its parameters is *non-regular*: instead of getting a normal limit and convergence rate $\sqrt{n}$ as usual, we get a symmetric exponential limit and convergence rate n; see e.g. IS II. This is typical of situations, as here, where the *support* (smallest set carrying full probability, 1) depends on the parameters.

2. *Exponential* $E(\lambda)$, $\lambda > 0$. This has density

$$f(x) = \lambda e^{-\lambda x} \qquad (x \ge 0), \qquad 0 \qquad (x < 0)$$

and distribution

$$F(x) = 1 - e^{-\lambda x} \qquad (x \ge 0), \qquad 0 \qquad (x \le 0).$$

Here the mean is $E[X] = 1/\lambda$. MLE is regular, and the MLE $\hat{\lambda} = 1/\bar{x}$, as one would expect (sample mean $\bar{x}$ corresponds to population mean $1/\lambda$).

3. *Normal* $N(\mu, \sigma^2)$; μ real, $\sigma > 0$. Here the density is

$$f(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}(x - \mu)^2/\sigma^2\}.$$

This is a density, and (as the notation suggests) it does indeed have mean μ and variance σ^2 [II.3 Example 1a, L7].

The case $\mu = 0, \sigma = 1$, the *standard normal* distribution $N(0,1)$, is so important it has special notation: the density and distribution function are written

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\},$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-\frac{1}{2}u^2\}du.$$

Note that $\Phi(0) = \frac{1}{2}$ by symmetry (and $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$); for other values, we have to use tables.

The MLEs for the population mean and variance are the sample mean and variance:

$$\hat{\mu} = \bar{X}, \qquad \hat{\sigma^2} = \bar{S}^2 \ (:= \frac{1}{n}\sum_{1}^{n}(X_k - \bar{X}^2)).$$

Note that we use the "$1/n$" definition of the sample variance (so that "bar, or average, corresponds to expectation", rather than the alternative "$1/(n-1)$" definition (to get the sample variance unbiased). Always check!

4. *Chi-square with n degrees of freedom (df)*, $\chi^2(n)$. This is the distribution of $X_1^2 + \ldots + X_n^2$, where $X_1, \ldots, X_n$ are independent and identically distributed (iid) $N(0,1)$. It has density

$$f(x) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.x^{\frac{1}{2}n-1}\exp\{-\frac{1}{2}x\} \qquad (x > 0),$$

mean n and variance $2n$; see e.g. [BF], §2.1.

We quote (see e.g. [BF], Th. 2.4):

(i) $\bar{X}$ and S^2 are independent; (ii) $\bar{X} \sim N(\mu, \sigma^2/n)$; (iii) $nS^2/\sigma^2 \sim \chi^2(n-1)$.

5. *Student t-distribution with n df*, $t(n)$. This is defined as the distribution of

$$X := \sqrt{n}U/\sqrt{V},$$

where $U \sim N(0,1)$, $V \sim \chi^2(n)$ and U, V are independent. It has distribution

$$f(x) = \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\sqrt{\pi n}\Gamma(\frac{1}{2}n)}\Big(1 + \frac{x^2}{n}\Big)^{-\frac{1}{2}(n+1)}.$$

By above,

$$\sqrt{n-1}(\bar{X}-\mu)/S \sim t(n-1).$$

This is very useful when estimating the mean μ without knowing the variance σ^2 (or standard deviation – SD – σ): the *nuisance parameter* σ cancels on forming the Student t ratio above.