pfsprob7.tex

PROBLEMS 7 21.11.2013

Q1 *Properties of conditional expectation.*
(i) For $\mathcal{B}$ the trivial σ-field $\{\emptyset, \Omega\}$, show that $E[Y|\mathcal{B}] = E[Y]$. (Interpretation: a conditional expectation knowing nothing is the same as an unconditional expectation.)
(ii) For $\mathcal{B}$ the whole σ-field $\mathcal{A}$, show that $E[Y|\mathcal{B}] = Y$. (Interpretation: given the whole σ-field $\mathcal{A}$, we know everything, so no randomness remains to average over, so taking the conditional expectation changes nothing.)
(iii) If Y is $\mathcal{B}$-measurable, show that $E[Y|\mathcal{B}] = Y$.
(iv) (Tower property). Show that if $\mathcal{C} \subset \mathcal{B}$, then $E[E[Y|\mathcal{B}]\ |\mathcal{C}] = E[Y|\mathcal{C}]$.
(iv') (Tower property'). Show that if $\mathcal{C} \subset \mathcal{B}$, then $E[E[Y|\mathcal{C}]\ |\mathcal{B}] = E[Y|\mathcal{C}]$.

Q2 *Time-inversion for Brownian motion.* For B Brownian motion, and X defined for $t \neq 0$ by

$$X_t := tB(1/t),$$

show that X is again Brownian motion (we say X is obtained from B by *time-inversion*).

Hence or otherwise show that

$$B(t)/t \to 0 \qquad a.s. \qquad (t \to \infty).$$

Q3. *Brownian bridge* X is defined for $t \in [0,1])$ and B BM by

$$X_t := B_t - tB_1.$$

Show that X is Gaussian with mean 0 and covariance $\min(s,t) - st$.

Q4 (*Hypergeometric distribution*). Recall that $\binom{n}{k}$ is the number of subsets of size k of a st of size n, so

$$\sum_k \binom{n}{k} = 2^n$$

decomposes the total number 2^n of subsets of a set of size n by size k. Recall also that $\binom{n}{k}$ counts the number of (downward) paths from the vertex (the

'1' at the top) to the entry $\binom{n}{k}$ in Pascal's triangle. Show that

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n},$$

(i) by decomposing the number of subsets of size n of a set of $2n$ balls, n white and n black, according to how many white balls they contain;
(ii) by equating coefficients of x^n in the identity $(1+x)^{2n} \equiv (1+x).(1+x)^n$;
(iii) by counting routes from the vertex to the central entry $\binom{2n}{n}$ in row $2n$, according to where they cross row n.
[We shall use Q3 in the study of the Bernoulli-Laplace distribution – see Problems 10 Q1.] NHB