pfsprob9.tex

PROBLEMS 9 5.12.2013

Q1 (*Feller relation*). Start a Markov chain in state 0 at time 0. By the Markov property, the chain *regenerates* on its returns to 0. The chain is in 0 at time 0 by definition; on its subsequent returns to 0 (at *positive* times), we say that the chain is at 0 for the *first time*, the second time etc. Write

$$n_n := P(\text{chain is at } 0) \qquad (u_0 = 1; n = 0, 1, 2, \ldots),$$

$$f_n := P(\text{chain is at 0 for the first time}) \qquad (n = 1, 2, \ldots)$$

((u_n) is called a *renewal sequence*, with *first-return distribution* (f_n)), $U(s)$, $F(s)$ for the GFs

$$U(s) := \sum_{n=0}^{\infty} u_n s^n, \qquad F(s) := \sum_{n=1}^{\infty} f_n s^n.$$

Obtain the *Feller relation*

$$U(s) = 1/(1 - F(s)).$$

Q2 (*Ehrenfest urn*). Show that the Ehrenfest urn has no limit distribution, but has an invariant distribution the *binomial distribution* $\pi = (\pi_j)$, where

$$\pi_j = 2^{-d}\binom{d}{j}.$$

We say that the chain satisfies the *detailed balance* condition if there exists a distribution $\pi = (\pi_j)$ with $\pi_j > 0$ for all j and

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \text{for all } i,\, j. \qquad (DB)$$

The chain is *reversible* if its probabilistic structure is invariant under time-reversal (the chain looks the same if run backwards in time). We quote (Kolmogorov's theorem, of 1936) that reversibility and detailed balance are equivalent, and that then π in (DB) is the unique invariant distribution.

Q3 (*Ehrenfest urn*, continued). Show that the Ehrenfest urn is reversible, and derive its invariant distribution from detailed balance. NHB