pfssoln6.tex

SOLUTIONS 6 21.11.2013

Q1. (i)
$$EM(t) = 1$$
, as $N(0, 1)$ has MGF $e^{\frac{1}{2}s^2}$ and $N(0, t) =_d \sqrt{t} \cdot N(0, 1)$.

$$E[M_t | \mathcal{F}_u] = E[\exp\{s[B_u + (B_t - B_u)] - \frac{1}{2}ts^2\} | \mathcal{F}_u]$$

=
$$\exp\{sB_u - \frac{1}{2}us^2\} \cdot E[\exp\{s(B_t - B_u) - \frac{1}{2}(t - u)s^2\} | \mathcal{F}_u],$$

taking out what is known. The first term on RHS is M_u . Using the Strong Markov Property for BM to start afresh at time u, the second is $E[M_{t-u}]$, which is 1 by above. So M is a mg.

(ii) The stopping time T_n is bounded, so Doob's Stopping Time Principle gives $E[M(T_n)] = 1$:

$$1 = E \exp\{sB(T_n) - T_n \cdot \frac{1}{2}s^2\}$$

= $E[\exp\{sB(n) - n \cdot \frac{1}{2}s^2\}I(\tau > n)] + E[\exp\{sB(\tau_t) - \tau_t \cdot \frac{1}{2}s^2\}I(\tau \le n)].$

On $\tau > n$, B(n) < t, so the first term on RHS is at most

$$\exp\{st - n \cdot \frac{1}{2}s^2\} \cdot P(\tau > n) \le e^{st} \cdot P(\tau > n) \to 0 \qquad (n \to \infty).$$

Letting $n \to \infty$, $1 = E[\exp\{st - \tau_t \cdot \frac{1}{2}s^2\} \cdot I(\tau_t < \infty)]$. But $\tau_t < \infty$ a.s. (otherwise B_u would lie below t for all u, so would have support bounded above; but each B_u is normal, so has support the whole line). So

$$1 = E[\exp\{st - \tau_t \cdot \frac{1}{2}s^2\}]: \quad E[\exp\{-\tau_t \cdot \frac{1}{2}s^2\}] = e^{-st}, \quad E[\exp\{-s\tau_t\}] = e^{-t\sqrt{2s}}.$$

(iii) The first-passage process τ is non-decreasing, as it takes longer to reach a higher level. Using the Strong Markov Property at time τ_t shows that the further time to first passage to level t + u is independent of \mathcal{F}_t , and so of τ_t ; this says that the process τ has independent increments. This further time has the same distribution as τ_u , by the stationary-increments property of BM; so τ has stationary increments. so τ is a non-decreasing Lévy process, i.e. a subordinator.

(iv) By (iii),
$$E \exp\{-s\tau_{ct}\} = \exp\{-t.c\sqrt{2s}\} = \exp\{-t.\sqrt{2sc^2}\} = E \exp\{-sc^2\tau_t\}$$

Comparing, $c^2 \tau_t =_d \tau_{ct}$: $\tau_t =_d \tau_{ct}/c^2$.

Q2.

$$\phi(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-sx - \frac{1}{2x}) \cdot \frac{dx}{x^{3/2}}$$

Differentiate under the integral sign (as we may, the integrand being monotone in s – we quote this):

$$\phi'(s) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-sx - \frac{1}{2x}) \cdot \frac{dx}{\sqrt{x}}$$

The change of variable suggested interchanges the two terms in the exponential. It reverses the limits, and (check)

$$\frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{2s}} \cdot \frac{du}{u^{3/2}}.$$

This gives

$$\phi'(s) = -\frac{1}{\sqrt{2s}} \phi(s) : \qquad \frac{\phi'(s)}{\phi(s)} = -\frac{1}{\sqrt{2s}}$$

Integrate: $\log \phi(s) = -\sqrt{2s} + c$, $\phi(s) = ce^{-\sqrt{2s}}$. But $\phi(0) = \int f = 1$, so c = 1 and $\phi(s) = e^{-\sqrt{2s}}$. //

Note. This is $L\acute{e}vy$'s density – a rare case (with the normal and Cauchy) where one can find a stable density explicitly.

Q3. Adding independent random variables multiplies Laplace transforms (as with CFs – from the Multiplication Theorem), so $X_1 + \ldots + X_n$ has Laplace transform $[\phi(s)]^n = e^{-n\sqrt{2s}}$. Replacing s by s/n^2 , $X_1 + \ldots + X_n)/n^2$ has Laplace transform $\phi(s) = e^{-\sqrt{2s}}$, the Laplace transform of X. So $(X_1 + \ldots + X_n)/n^2$ has the same distribution as X, as required.

This does not contradict the SLLN, as X has infinite mean.

Q4. $|\int_B f_n - \int_B f| = |\int_B (f_n - f| \le \int_B |f_n - f|$. Taking sups over B proves the inequality. Next, with $a \wedge b := \min(a, b)$, $|f_n - f| = f_n + f - 2f_n \wedge f$ (check). Integrate: $\int f_n = 1$, $\int f = 1$ as these are densities. As $0 \le f_n \wedge f \le f$, integrable, dominated convergence gives $\int f_n \wedge f \to \int f = 1$. So the integral of RHS $\to 1+1-2 = 0$. So the integral of LHS $\to 0$ also. // NHB