pfssoln9.tex

SOLUTION 9 12.12.2013

Q1. $u_0 = 1$ by definition; for $n \geq 1$,

$$u_n = P(\text{in } 0 \text{ at time } n) = \sum_{k=1}^{n} P(\text{in } 0 \text{ at time } k \text{ for the first time and at time } n)$$

$$= \sum_{k=1}^{n} P(\text{in } 0 \text{ at time } n \mid \text{in } 0 \text{ at time } k \text{ for the first time}) P(\text{in } 0 \text{ at time } k \text{ for the first time})$$

$$= \sum_{k=1}^{n} u_{n-k} f_k,$$

using the Markov property to restart the chain from 0 at time k. So the GF is

$$U(s) := \sum_{n=0}^{\infty} u_n s^n = 1 + \sum_{n=1}^{\infty} s^n \sum_{k=1}^{n} u_{n-k} f_k.$$

Put $j := n - k$; as $1 \leq k \leq n < \infty$, the new limits on summation are $0 \leq j < \infty, 1 \leq k \leq \infty$. We obtain

$$U(s) = 1 + \sum_{j=0}^{\infty} u_j s^j \sum_{k=1}^{\infty} f_k s^k = 1 + U(s)F(s),$$

so $U(s) - U(s)F(s) = 1$:

$$U(s) = 1/(1 - F(s)).$$

Q2.

$$p_{i,i-1} = \frac{i}{d}, \qquad p_{i,i+1} = \frac{d-i}{d}, \qquad \pi_i = 2^{-d}\binom{d}{i}.$$

Now

$$\binom{d}{j-1}(d-j+1) = \frac{d!}{(j-1)!(d-j+1)!}.(d-j+1) = \frac{d!}{(j-1)!(d-j)!} = \binom{d}{j}.j, \qquad (i)$$

$$\binom{d}{j+1}(j+1) = \frac{d!}{(j+1)!(d-j-1)!}(j+1) = \frac{d!}{j!(d-j-1)!} = \binom{d}{j}.(d-j). \qquad (ii)$$

So by (i) and (ii),

$$\begin{aligned}
(\pi P)_j &= \sum_i \pi_i p_{ij} \\
&= \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j} \\
&= 2^{-d} \binom{d}{j-1} \frac{(d-j+1)}{d} + 2^{-d} \binom{d}{j+1} \frac{(j+1)}{d} \\
&= \frac{2^{-d}}{d} \binom{d}{j} \{j + (d-j)\} = 2^{-d} \binom{d}{j} \\
&= \pi_j .
\end{aligned}$$

So $\pi P = \pi$, and π is invariant, as required. //

Q3. With π as in Q1,

$$\pi_i p_{i,i+1} = 2^{-d} \binom{d}{i} . \frac{d-i}{d} = 2^{-d} \frac{d!}{(d-i)!i!} . \frac{d-i}{d} = 2^{-d} \binom{d-1}{i},$$

and similarly

$$\pi_{i+1} p_{i+1,i} = 2^{-d} \binom{d-1}{i},$$

proving detailed balance, and so reversibility. Assuming reversibility, we can use detailed balance to calculate the invariant distribution:

$$i = 0 : \qquad \pi_1 = \pi_0 \frac{p_{01}}{p_{10}} = \frac{\pi_0}{\frac{1}{d}}.$$

$$i = 1 : \qquad \pi_2 = \pi_1 \frac{p_{12}}{p_{21}} = \frac{\pi_0}{\frac{1}{d}} . \frac{1 - \frac{1}{d}}{\frac{2}{d}}, \ldots,$$

$$\pi_i = \frac{\pi_0}{\frac{1}{d}} . \frac{1 - \frac{1}{d}}{\frac{2}{d}} . \ldots . \frac{1 - \frac{i-1}{d}}{\frac{i}{d}} = \pi_0 . \frac{d(d-1) \ldots (d-i+1)}{1.2 \ldots .i} = \pi_0 \binom{d}{i}.$$

Then $\sum_i \pi_i = 1$ gives

$$\pi_0 \sum_i \binom{d}{i} = \pi_0 . 2^d = 1, \qquad \pi_0 = 2^{-d}, \qquad \pi_i = 2^{-d} \binom{d}{i},$$

as before.

NHB