pfsl21(13).tex

Lecture 21. 27.11.2013 (half-hour – Problems)

Note. Relevant here is the concept of *entropy* – a measure of disorder. For details, see Problems and Solutions 9 and 10.

Markov Chain Monte Carlo (MCMC).

The area of *Markov chain Monte Carlo (MCMC)*, for which see e.g. Häggström [Hag] Ch. 7, 8, originated in physics, but has since become extremely important in statistics, particularly Bayesian statistics (for which see e.g. SMF, IV). The idea is to sample, or simulate, from a distribution π. If this is straightforward, fine (see e.g. IS II for simulation) – but it may not be. In this case, the method of MCMC is to find a Markov chain $X = (X_n)$ with π as its limit distribution. Then we can run the chain, knowing that its distribution for large n will approximate π. How long we have to wait for the approximation to be good enough for our purposes depends on the transition matrix P of the chain – and in particular, on its spectral gap.

Note. The two most important developments in Statistics in recent decades have been MCMC and wavelets.

4. Finite and infinite chains

Finite chains have special and useful properties.

Theorem. For a finite Markov chain, it is impossible for all states to be transient: a finite chain must contain at least one persistent state.

Proof. If the state-space is $\{1, \ldots, N\}$, for each i and each n

$$1 = \sum_{j=1}^{N} p_{ij}(n). \qquad (a)$$

Let $n \to \infty$: if j is transient, the total expected time in it is finite: $\sum_n p_{ij}(n) < \infty$. So

$$p_{ij}(n) \to 0 \qquad (n \to \infty). \qquad (b)$$

If *all* states were transient, then letting $n \to \infty$ in (a) and using (b) would give the contradiction $1 = 0$. So not all states in a finite chain can be transient. //

Note. 1. An infinite chain can easily consist of only transient states. A trivial example is walk to the right on the integers: $p_{i,i+1} = 1$, with the other p_{ij} 0.

A non-trivial example is given by Pólya's theorem: simple symmetric random walk on the integer lattice $\mathbb{Z}^d$ is transient for $d \geq 3$ (but recurrent for $d = 1, 2$). See e.g. [F], XIV.7, [GS], 13.11 p.560.
2. The sum $\sum_n p_{ij}(n)$ is the expected total time spent in state j, starting from i. With only finitely many states, and infinite total time altogether, at least one of these sums must thus be infinite.

Theorem. A persistent state j in a finite chain is positive (= non-null).

Proof. If the finite chain has state-space $\{1, \ldots, N\}$, assume there is a null state. Let C be the equivalence class containing it. Since C is closed, we can consider the subchain induced on C. Then

$$1 = \sum_{k \in C} p_{ik}(n) \qquad \text{(finite sum)}.$$

Let $n \to \infty$: each $p_{ik}(n) \to 0$, so the sum on the RHS $\to 0$, giving $1 = 0$. This contradiction gives the non-existence of null states in a finite chain. //

The restriction to *finite* chains is essential here: e.g., simple symmetric random walk on the integers has all states persistent null.

The limit theorem above is due to Kolmogorov in 1936. The algebraic treatment we have given is in terms of matrices – and in the case of an infinite chain, these will be infinite matrices. Dealing with infinite rather than finite matrices is possible (with care, and under suitable conditions) – but belongs to Functional Analysis rather than to Linear Algebra. Infinite-dimensional versions of the Perron-Frobenius theorem exist, such as the *Krein-Rutman theorem* for positive operators. But this leads beyond the scope of this course.

Continuous state-space

It turns out that, although the language of matrices is so useful in the above, one can extend much of the treatment above to situations where the state space is *continuous* rather than discrete. It turns out also that it is this case that is most useful in applications, particularly MCMC. For a full treatment, see e.g. Meyn & Tweedie [MT]. Much of the theory above extends to the continuous-state case. Again the transience-recurrence dichotomy is crucial, but there are now various possible types of recurrence. One of the most important is *Harris recurrence* (T. E. HARRIS (1919-2005) in 1956).