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Lecture 24. 2.12.2014 (half-hour – Problems)

Martingale convergence

One reason why martingales (mgs) are so useful is that they have good convergence properties – under suitable conditions. We state some of the key results, without proof; for details, see e.g. SP, L18-19.

Call $X = (X_n)$ L_1-*bounded* if $\sup_n E[|X_n|] < \infty$, i.e.

$$E[|X_n|] \leq K \quad \text{for all } n,$$

for some constant K.

Doob's (Sub-)Martingale Convergence Theorem. An L_1-bounded (sub)martingale is a.s. convergent.

The proof depends on Doob's Upcrossing Inequality (see e.g. SP L18).

Uniform integrability (UI). Call X_n *uniformly integrable* (UI) if

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \to 0 \qquad (a \to \infty).$$

If the index set $\{1, 2, \ldots\}$ of the filtration $(\mathcal{F}_n)$ extends to $\{1, 2, \ldots, \infty\}$ so that $\{X_n : n = 1, 2, \ldots, \infty\}$ is a (sub-)mg w.r.t. this filtration, the (sub-)mg is called *closed*, with *closing* (or *last*) element X_∞.

Theorem. Let (X_n) be a UI submg. Then $\sup_n E[X_n^+] < \infty$, and X_n converges to a limit X_∞ a.s. and in L_1, which closes the submg: $X = (X_n)$ is a closed submg, closed by X_∞.

Theorem. X_n is a UI mg iff X_n is a closed mg iff there exists $Y \in L_1$ with

$$X_n = E[Y|\mathcal{F}_n].$$

Then $X_n \to E[Y|\mathcal{F}_\infty]$ a.s. and in L_1.

Corollary (UI Mg Convergence Theorem). For a mg $X = (X_n)$, the following are equivalent:
(i) X is UI;

(ii) X converges a.s. and in L_1 (to X_∞, say);
(iii) X is closed by a random variable Y: $X_n = E[Y|\mathcal{F}_n]$;
(iv) X is closed by its limit X_∞: $X_n = E[X_\infty|\mathcal{F}_n]$.

Note. 1. The UI mgs – equivalently by above the closed mgs – (also called *regular* mgs) are the 'nice' mgs. Note that all the randomness is in the closing rv $Y = X_\infty$. As time progresses, more of Y is revealed as more information becomes available. (Think of progressive revelation, as in – choose your metaphor – a 'striptease', or, 'the Day of Judgement'.)
2. UI (or closed) mgs are also common, and crucially important in Mathematical Finance. There, one does two things: (i) *discount* all asset prices (so as to work with real rather than nominal prices); (ii) change from the real-world probability measure P to an equivalent martingale measure Q (EMM, or *risk-neutral measure*) under which discounted asset prices $\tilde{S}_t$ become (Q)-mgs:

$$\tilde{S}_t = E_Q[\tilde{S}_T|\mathcal{F}_t]$$

($T < \infty$ is e.g. the expiry time of an option). See e.g. [BK], esp. Ch. 4.

Matters are simpler in the L_p case for $p \in (1, \infty)$. Call $X = (X_n)$ L_p-*bounded* if

$$\sup_n \|X_n\|_p < \infty$$

(so in particular each $X_n \in L_p$). We may take $p = 2$ for simplicity, and because of the link with Hilbert-space methods and the important *Kunita-Watanabe Inequalities.* We quote (for proof see e.g. SP L19)

Theorem (L_p-Mg Theorem). If $p > 1$, an L_p-bounded mg X_n is UI, and converges to its limit X_∞ a.s. and in L_p.

3. Martingales in continuous time

A stochastic process $X = (X(t))_{0 \le t < \infty}$ is a *martingale* (mg) relative to $(\{\mathcal{F}_t\}, P)$ if
(i) X is adapted, and $E|X(t)| < \infty$ for all $\le t < \infty$;
(ii) $E[X(t)|\mathcal{F}_s] = X(s)$ P- a.s. $(0 \le s \le t)$,
and similarly for submgs (with $\le$ above) and supermgs (with $\ge$).

In continuous time there are regularization results, under which one can take $X(t)$ RCLL in t (basically $t \to EX(t)$ has to be right-continuous). Then the analogues of most results for discrete-time martingales hold true.