pfsprob7.tex

PROBLEMS 7 25.11.2014

Q1 (*Hypergeometric distribution*). Recall that $\binom{n}{k}$ is the number of subsets of size k of a st of size n, so

$$\sum_{k} \binom{n}{k} = 2^n$$

decomposes the total number 2^n of subsets of a set of size n by size k. Recall also that $\binom{n}{k}$ counts the number of (downward) paths from the vertex (the '1' at the top) to the entry $\binom{n}{k}$ in Pascal's triangle. Show that

$$\sum_{k} \binom{n}{k}^2 = \binom{2n}{n},$$

(i) by decomposing the number of subsets of size n of a set of 2n balls, n white and n black, according to how many white balls they contain; (ii) by equating coefficients of x^n in the identity $(1+x)^{2n} \equiv (1+x).(1+x)^n$; (iii) by counting routes from the vertex to the central entry $\binom{2n}{n}$ in row 2n,

according to where they cross row n.

Q2 (*Bernoulli-Laplace urn model*). By Q1 or otherwise, show that the Bernoulli-Laplace urn is

(i) reversible/has detailed balance;

(ii) has invariant distribution the hypergeometric distribution

$$\pi_i = \binom{d}{i}^2 / \binom{2d}{d}$$

(this is a probability distribution, by Q1).

Q3 (*Bernoulli-Laplace urn* continued). Show that the mean recurrence time of state 0 is

$$\mu_0 = \binom{2d}{d} \sim 4^d / \sqrt{\pi d} \qquad (d \to \infty).$$

Interpret this in the context of Statistical Mechanics, where d is of the order of Avogadro's number, 6.02×10^{23} .

Q4 Branching processes. In a population model, one starts with a single ancestor (the 0th generation). On death, he is replaced by a random number Z of offspring, with PGF P(s) (the 1st generation), and mean $\mu = E[Z]$. They reproduce independently and in the same way, their offspring forming the second generation, with PGF P_2 , and so on. Show that:

(i) $P_2(s) = P(P(s))$, the second (functional) iterate of P;

(ii) the *n*th generation, of size Z_n say, has PGF P_n , the *n*th functional iterate of P (defined inductively by $P_n = P_{n-1}(P) = P(P_{n-1})$);

(iii) the mean generation size is $E[Z_n] = \mu^n$.

NHB