pfssoln2.tex

SOLUTIONS 2. 28.10.2014

Q1. (i) The joint density of the x_i is

$$f(x) = (2\pi)^{-\frac{1}{2}n} \prod_{i=1}^{n} \exp\{-\frac{1}{2}x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\sum_1^n x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\|x\|^2\}.$$

The Jacobian of the change of variable is the determinant $|O|$, $= 1$ as O is orthogonal ($=$ length-preserving). So the joint density of the y_i is

$$g(y) = (2\pi)^{-\frac{1}{2}n} \exp\{-\|y\|^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\sum_1^n y_i^2\},$$

which says that the y_i are iid $N(0,1)$.
(ii) The condition for a matrix O to be orthogonal is that the rows are of length 1 and orthogonal vectors. Take the first row as e_1, and use Gram-Schmidt orthogonalisation to find e_2 orthogonal to e_1, then e_3 orthogonal to e_1, e_2 etc. The e_i form the rows of an orthogonal matrix with first row e_1.
(iii) Put $Z_i := (X_i - \mu)/\sigma$, $Z := (Z_1, \ldots, Z_n)^T$; then the Z_i are iid $N(0,1)$,

$$\bar{Z} = (\bar{X} - \mu)/\sigma, \qquad nS^2/\sigma^2 = \sum\nolimits_1^n (Z_i - \bar{Z})^2.$$

Also

$$\sum\nolimits_1^n Z_i^2 = \sum\nolimits_1^n (Z_i - \bar{Z})^2 + n\bar{Z}^2,$$

since $\sum_1^n Z_i = n\bar{Z}$. The terms on the right above are quadratic forms, with matrices A, B say, so we can write

$$\sum\nolimits_1^n Z_i^2 = Z^T AZ + Z^T BX. \qquad (*)$$

Put $W := PZ$ with P a Helmert transformation with first row $(1, \ldots, 1)/\sqrt{n}$:

$$W_1 = \frac{1}{\sqrt{n}} \sum\nolimits_1^n Z_i = \sqrt{n}\bar{Z}; \qquad W_1^2 = n\bar{Z}^2 = Z^T BZ.$$

So by above,

$$Z^T AZ = \sum\nolimits_1^n (Z_i - \bar{Z})^2 = nS^2/\sigma^2, = \sum\nolimits_2^n W_i^2,$$

as $\sum_1^n Z_i^2 = \sum_1^n W_i^2$. But the W_i are independent (by the orthogonality of P), so W_1 is independent of $W_2, \ldots, W_n$. So W_1^2 is independent of $\sum_2^n W_i^2$. So nS^2/σ^2 is independent of $n(\bar{X}-\mu)^2/\sigma^2$, so S^2 is independent of $\bar{X}$, as claimed.

Q2. (i) For $n = 1$, the mean is 1, because a $\chi^2(1)$ is the square of a standard normal, and a standard normal has mean 0 and variance 1. The variance is 2, because the fourth moment of a standard normal X is 3, and

$$var(X^2) = E[(X^2)^2] - [E(X^2)]^2 = 3 - 1 = 2.$$

For general n, the mean is n because means add, and the variance is $2n$ because variances add over independent summands.
(ii) For X standard normal, the MGF of its square X^2 is

$$M(t) := \int e^{tx^2}\phi(x)dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{tx^2}.e^{-\frac{1}{2}x^2}dx.$$

We see that the integral converges only for $t < \frac{1}{2}$, when it is $1/\sqrt{(1-2t)}$:

$$M(t) = 1/\sqrt{1-2t} \qquad (t < \frac{1}{2}) \qquad \text{for } X\ N(0,1).$$

So by definition of $\chi^2(n$, the MGF of a $\chi^2(n)$ is

$$M(t) = 1/(1-2t)^{\frac{1}{2}n} \qquad (t < \frac{1}{2}) \qquad \text{for } X\ \chi^2(n).$$

(iii) First, $f(.)$ is a density, as it is non-negative, and integrates to 1:

$$\begin{aligned}\int f(x)dx &= \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x)dx \\ &= \frac{1}{\Gamma(\frac{1}{2}n)}.\int_0^\infty u^{\frac{1}{2}n-1}\exp(-u)du \qquad (u := \frac{1}{2}x) \\ &= 1,\end{aligned}$$

by definition of the Gamma function. Its MGF is

$$M(t) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty e^{tx}.x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x)dx = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x(1-2t))dx.$$

Substitute $u := \frac{1}{2}x(1-2t)$ in the integral. One obtains

$$M(t) = (1-2t)^{-\frac{1}{2}n}.\frac{1}{\Gamma(\frac{1}{2}n)}.\int_0^\infty u^{\frac{1}{2}n-1}e^{-u}du = (1-2t)^{-\frac{1}{2}n},$$

by definition of the Gamma function. // NHB