pfssoln4.tex

SOLUTIONS 4. 11.11.2014

Q1. (i) This is linearity of expectation E.
(ii) Variance $var\ X := E[(X - EX)^2]$ is translation-invariant (by (i)), and depends quadratically on dilations a.
(iii) Take square roots of (ii).
(iv) Covariance $cov\ X := E[(X - EX)(Y - EY)]$ is translation-invariant (by (i), as in (ii)), and again depends quadratically on dilation a, as in (ii).
(v) $corr(X, Y) := cov(X, Y)/\sqrt{var(X).var(Y)}$, so the a^2 cancel: correlation is invariant under affine transformations (i.e., under changes of location and scale). If $\rho(X, Y) := corr(X, Y)$, ρ ($\in [-1, 1]$, by the Cauchy-Schwarz inequality – Problems 3 Q4) is a measure of the *linkage*, or *association*, between X and Y – i.e. of the strength of their dependence.

Sample versions:
(i) $\overline{aX + b} = a\overline{X} + b$ [linearity of averaging].
(ii) Using $S_X^2 := \overline{[(X - \overline{X})^2]}$ for the sample variance of X: $S_{aX+b}^2 = a^2 S_X^2$ [as in (ii)].
(iii) Take square roots: $S_{aX+b} = |a|.S_X$.
(iv) $S_{aX+b,aY+b} = a^2 S_{X,Y}$.
(v) Using $r(X, Y) := S_{X,Y}/(S_X.S_Y)$ for sample correlation: $r(aX + b, aY + b) = r(X, Y)$. Thus sample correlation too is affine-invariant (and $r \in [-1, 1]$, by Problems 3 Q3).

Q2. By SLLN applied to X^k,

$$\hat{\mu}_k := \overline{X^k} \to E[X^k] = \mu_k \quad a.s. \quad (n \to \infty).$$

The kth *central moment* is $\hat{\mu}_k^0 := \overline{(X - \overline{X})^k}$. Then

$$\begin{aligned}\hat{\mu}_k^0 := \overline{(X - \overline{X})^k} &= \overline{\sum\nolimits_0^k \binom{k}{i} X^i (-)^{k-i} (\overline{X})^{k-i}} \\ &= \sum\nolimits_0^k \binom{k}{i} (\overline{X^i}) (-)^{k-i} (\overline{X})^{k-i}.\end{aligned}$$

By SLLN, as $n \to \infty$ this tends a.s. to

$$\sum\nolimits_0^k \binom{k}{i} E[X^i](-)^{k-i}[EX]^{k-i} = \sum\nolimits_0^n E[(X - EX)^k] = \mu_k^0. \qquad //$$

Q3.

$$\log(1 + \mu t + \frac{1}{2}\mu_2 t^2 + \ldots) = \mu t + \frac{1}{2}\mu_2 t^2 + \ldots - \frac{1}{2}(\mu t + \ldots)^2 + \ldots$$

$$= \mu t + \frac{1}{2}(\mu_2 - \mu^2)t^2 + \ldots = \mu t + \frac{1}{2}\sigma^2 t^2 + \ldots.$$

Equating coefficients of t gives $\kappa_1 = \mu$. Centring at the mean multiplies the MGF $M(t)$ by $e^{-\mu t}$, so subtracts μt from the CGF, so leaves coefficients of powers of t^k unchanged for $k \geq 2$, so $\kappa_k = \kappa_{k,0}$ for $k \geq 2$, giving (i). Equating coefficients of t^2 gives $\kappa_2 = \sigma^2$, giving (ii). We can now take $\mu = 0$ without loss of generality, so

$$M(t) = 1 + \frac{1}{2}\sigma^2 t^2 + \frac{1}{6}\mu_3^0 t^3 + \frac{1}{24}\mu_4^0 t^4 + \ldots$$

Take logs and use $\log(1 + x) = x - \frac{1}{2}x^2 + \ldots$:

$$K(t) = \log M(t) = \frac{1}{2}\sigma^2 t^2 + \frac{1}{6}\mu_3^0 t^3 + \frac{1}{24}\mu_4^0 t^4 + \ldots - \frac{1}{2}[\frac{1}{2}\sigma^2 t^2 \ldots]^2.$$

As $K(t) = \sum \kappa_k t^k / k!$, equating coefficients of t^3 gives $\kappa_3 = \mu_3^0$, which is (iii). Equating coefficients of t^4 gives $\kappa_4 = \mu_k^0 - 3\sigma^4$ (3 8s are $24 = 4!$), which is (iv). Finally, X is normal $N(\mu, \sigma)$ iff $M(t) = e^{\mu t + \sigma^2 t^2/2}$ iff $K(t) = \mu t + \sigma^2 t^2/2$ iff all cumulants higher than the second vanish.

Q4. (i) Both μ_3^0 and $\sigma^2 = \mu_2^0$ are translation-invariant, by Q1. Under dilation by a, μ_3^0 scales by a^3, σ scales by a, so γ_1 is invariant as the a^3 cancels. Similarly, γ_2 is invariant as the a^4 cancels. Then (ii) is Q2, and (iii) follows from (ii) by SLLN.
Note. 1. Skewness, or asymmetry, is common in financial data. One reason for this is psychological: a given profit gives pleasure, but not as much as the pain of an equal loss. Similarly, volatility increases after a price fall, so one can typically recover the arrow of time from a time series of prices. Discounted prices may be stationary, but will generally not be time-reversible.
2. Financial data typically have much fatter tails than normal (and this shows up in the kurtosis). This has vital implications for risk management (as is more widely appreciated now than before the Crash of 2007-8).

NHB