pfssoln7.tex

SOLUTIONS 7 2.12.2014

Q1: *Hypergeometric distribution.* (i) The number of subsets of size n of a set of size $2n$ is $\binom{2n}{n}$. If this subset contains k white balls, these can be chosen in $\binom{n}{k}$ ways; the remaining $n-k$ balls are black, and can be chosen in $\binom{n}{n-k} = \binom{n}{k}$ ways, giving $\binom{n}{k}^2$ ways altogether; sum over k.
(ii)

$$\sum_i \binom{2n}{i} x^i = (\sum_j \binom{n}{j} x^j)(\sum_k \binom{n}{k} x^k).$$

Extracting the coefficient of x^n gives $\binom{2n}{n}$ on the left and $\sum_j \binom{n}{j}\binom{n}{n-j} = \sum_j \binom{n}{j}^2$ on the right.
(iii) The number of routes from the vertex to the central element in row $2n$ is $\binom{2n}{n}$. There are $\binom{n}{k}$ routes from the vertex to the element $\binom{n}{k}$ in row n. By symmetry of the ”square” with top corner the vertex and bottom corner $\binom{2n}{n}$ about its horizontal diagonal, the number of routes from $\binom{n}{k}$ to $\binom{2n}{n}$ is $\binom{n}{k}$. So there are $\binom{n}{k}^2$ routes passing through $\binom{n}{k}$; sum over k.

Q2: *Bernoulli-Laplace urn.* With π the hypergeometric distribution given (this is a probability distribution, by Q1),

$$\pi_i p_{i,i+1} = \frac{1}{\binom{2d}{d}} \binom{d}{i}^2 . (\frac{d-i}{d})^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2,$$

and similarly

$$\pi_{i+1} p_{i+1,i} = \frac{1}{\binom{2d}{d}} \binom{d}{i+1}^2 . (\frac{i+1}{d})^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2,$$

proving detailed balance, and so reversibility. Assuming reversibility, we can use detailed balance to calculate the invariant distribution:

$$\pi_i = \frac{\pi_0}{\left(\frac{1}{d}\right)^2} . \frac{\left(1-\frac{1}{d}\right)^2}{\left(\frac{2}{d}\right)^2} . \ldots . \frac{\left(1-\frac{i-1}{d}\right)^2}{\left(\frac{i}{d}\right)^2} = \pi_0 . \frac{(d(d-1)\ldots(d-i+1))^2}{(1.2.\ldots.i)^2} = \pi_0 \binom{d}{i}^2 .$$

Then $\sum_i \pi_i = 1$ gives

$$\pi_0 \sum_i \binom{d}{i}^2 = \pi_0 \binom{2d}{d}^2 = 1, \qquad \pi_0 = 1/\binom{2d}{d}, \qquad \pi_i = \binom{d}{i}^2 / \binom{2d}{d}. \quad //$$

Q3: *Bernoulli-Laplace urn (continued).* $\pi_i = 1/\mu_i$ by the Erdös-Feller-Pollard theorem (L19), so

$$\mu_0 = 1/\pi_0 = \binom{2d}{d}.$$

By Stirling's formula,

$$\mu_0 \sim \frac{\sqrt{2\pi} e^{-2d} 2d^{2d+\frac{1}{2}}}{(\sqrt{2\pi} e^{-d} d^{d+\frac{1}{2}})^2} = \frac{4^d}{\sqrt{\pi d}}.$$

Now as d is already very large (of the order of Avogadro's number 6×10^{23}), 4^d is astronomically vast – effectively infinite.

The interpretation of this in Statistical Mechanics is that μ_0 is the mean recurrence time of state 0, when all the $2d$ gas molecules are in one half of the container. Although this state is certain to recur (indeed, infinitely often), its mean recurrence time is so vast as to be effectively infinite – which explains why we do not see such states recurring in practice! This reconciles the theoretical reversibility of the model with the irreversible behaviour we observe when gases diffuse, etc. This was the Ehrenfests' motivatioon for their model, in 1912.

Note. Relevant here is the concept of *entropy* – a measure of disorder. This was introduced by Rudolf CLAUSIUS (1922-1888), in 1865, who formulated the Fist Law of Thermodynamics (Law of Conservation of Energy) and Second Law of Thermodynamics (entropy increases – things become more disordered):

1. Die Energie der Welt ist konstant (The energy of the world [the universe] is constant).

2. Die Entropie der Welt strebt einem Maximum zu (The entropy of the world [the universe] strives towards a maximum).

Q4: *Branching processes.*

(i) Z_2 is the sum of a random number, Z_1, of independent copies of Z. So

$$P_2(s) := E[s^{Z_2}] = \sum_{k=0}^{\infty} E[s^{Z_2} | Z_1 = k] P(Z_1 = k).$$

Now when $Z_1 = k$, Z_2 is a sum of k independent copies of Z, each with PGF $P(s)$, so has (conditional) PGF $P(s)^k$. So

$$P_2(s) = \sum_0^\infty p_k P(s)^k = P(P(s)).$$

(ii) Similarly, or by induction on n, Z_n has PGF P_n.
(iii)

$$P_n'(s) = P'(P_{n-1}(s)).P_{n-1}'(s).$$

So letting $s = 1$ ($R > 1$), or $s \uparrow 1$ ($R = 1$) and using Abel's Continuity Theorem, since P_{n-1}, being a PGF, has value 1 at 1, $P_n'(1) = P'(1).P_{n-1}'(1) = \mu.P_{n-1}'(1)$, so by induction

$$P_n'(1) = \mu^n : \quad E[Z_n] = \mu^n.$$

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