PROBABILITY FOR STATISTICS: EXAM 2014-15

Q1.(i) The *generating function* of a random variable X with non-negative integer values is $P(s)$, or $P_X(s)$, $:= \sum_{n=0}^{\infty} P(X = n)s^n$. Show that if X, Y are independent with generating functions P_X, P_Y, then $X + Y$ has generating function $P_{X+Y}(s) = P_X(s).P_Y(s)$.
(ii) A random variable X has the *Poisson* distribution with *parameter* λ, $X \sim P(\lambda)$, if $P(X = n) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, 2, \ldots$. Show that $P_X(s) = e^{-\lambda(1-s)}$.
(iii) If X, Y are independent, Poisson with parameters λ, μ, show that $X+Y$ is Poisson with parameter $\lambda + \mu$.
(iv) Show that $EX = \lambda$. Hence, given that $X + Y$ is Poisson, find its parameter without calculation.
(v) Define a *Poisson point process* with *parameter* λ, $Ppp(\lambda)$.
(vi) If $X = (X_t)$, $Y = (Y_t)$ are independent Poisson point processes with parameters λ, μ, show that $X + Y$ is a $Ppp(\lambda + \mu)$.

Q2. (i) Given a bivariate population (X, Y), define the *(population) correlation coefficient* ρ of X and Y. Given a sample $(X_1, Y_1), \cdots, (X_n, Y_n)$ from this population, define the *sample correlation coefficient* r.
(ii) Show that $-1 \leq \rho \leq 1$, $-1 \leq r \leq 1$.
(iii) Obtain the conditions for $\rho = \pm 1$, $r = \pm 1$.
(iv) Show that $r \to \rho$ a.s. as $n \to \infty$.

Q3.(i) Show that

$$\pi_i := \binom{d}{i}^2 \Big/ \binom{2d}{d} \qquad (i = 0, \cdots, d)$$

defines a probability distribution $\pi = (\pi_i)$ (the *hypergeometric distribution* $HG(d)$).
(ii) Define *detailed balance* of a Markov chain $P = (p_{ij})$ with respect to a distribution $\pi = (\pi_i)$.

State without proof what follows from detailed balance regarding reversibility and long-term behaviour.
(iii) There are $2d$ balls, d black and d white; these are divided between two urns, I and II, d balls to each urn. At each stage, a ball is chosen at random from each urn and the two are interchanged; the state of the system is the

number of black balls in the first urn. Write down the transition probability matrix $P = (p_{ij})$ of the Markov chain thus defined.
(iv) Show that the chain has detailed balance with respect to $HG(d)$.
(v) Find the recurrence time of state 0.
(vi) What is the physical importance of this model for large d?

Q4. (i) Define *standard Brownian motion* $B = (B(t))$.
(ii) Find the mean and covariance of B.
(iii) For each $c \in (0, \infty)$, show that if

$$B_c(t) := c^{-1} B(c^2 t),$$

$B_c := (B_c(t))$ is again a Brownian motion.
(iv) What does this tell us about the local (small-scale) behaviour of a Brownian path?
(v) Brownian bridge X is defined by $X(t) := B(t) - tB(1)$. Show that it is Gaussian, and find its mean and covariance.

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