PROBABILITY FOR STATISTICS: EXAMINATION 2015-16

Q1. Define the *Chi-square distribution* with n *degrees of freedom*, $\chi^2(n)$. Show that it has
(i) mean n, variance $2n$;
(ii) characteristic function $1/(1-2it)^{\frac{1}{2}n}$;
(iii) density

$$f(x) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x) \qquad (x > 0).$$

Q2. (i) Define the *cumulants* κ_n of a random variable.
(ii) The *skewness* and *kurtosis* are defined by ($\sigma^2 = \mu_{2,0}$)

$$\gamma_1 := \kappa_3/\kappa_2^{3/2} = \mu_{3,0}/\mu_{2,0}^{3/2} = \mu_{3,0}/\sigma^3, \quad \gamma_2 := \kappa_4/\kappa_2^2 = \frac{\mu_{4,0}}{\sigma^4} - 3 = \frac{\mu_{4,0}}{\mu_{2,0}^4} - 3.$$

Define their sample counterparts $\hat{\gamma}_1$, $\hat{\gamma}_2$.
(iii) Show that the sample central moments $\hat{\mu}_{k,0}$ tend to their population counterparts $\mu_{k,0}$ as the sample size increases (a.s.).
(iv) Show that also $\hat{\gamma}_1 \to \gamma_1$, $\hat{\gamma}_2 \to \gamma_2$.
(v) How does one recognise normality in terms of cumulants?
(vi) Show that skewness and kurtosis, sample and population versions, are *affine-invariant* (invariant under affine maps $x \mapsto ax + b$).
(vii) How might one use these results to test a population for normality?

Q3. Define the multivariate normal distribution.

Writing the multivariate normal distribution with mean vector μ and covariance matrix Σ as $N(\mu, \Sigma)$, state and prove Edgeworth's theorem giving the density $f(x)$ of $N(\mu, \Sigma)$. (You may quote any results you need on matrix theory, but these should be clearly stated.)

Q4. Prove the tower property of conditional expectations: that if $\mathcal{C} \subset \mathcal{B}$,

$$E[E[Y|\mathcal{B}]|\mathcal{C}] = E[Y|\mathcal{C}], \qquad E[E[Y|\mathcal{C}]|\mathcal{B}] = E[Y|\mathcal{C}].$$

Show that conditional expectation is idempotent, and so (being also linear) is a projection.

Prove the Conditional Mean Formula:

$$E[E[X|\mathcal{B}]] = E[X].$$

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