pfsl18(15).tex

Lecture 18. 23.11.2015 (half-hour – Problems)

Stationary distributions (continued).

Observe that the linear equations (SD) are homogeneous: if π is a solution, then so is $c\pi$ for any scalar c. We are only interested in solutions $\pi = (\pi_j)$ which are probability distributions, i.e. $\pi_j \geq 0$, $\sum_j \pi_j = 1$. There may well be solutions but *not* solutions of this type; we shall meet examples of this below.

Examples.

1. *Two states.* This is the simplest possible case:

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

There are two common interpretations:

(i) Motion on the line with constant speed,

$$\alpha = P(\text{change direction to left}|\text{going right}), \quad \beta = P(\text{change direction to right}|\text{going left}).$$

(ii) Rainfall. This chain has been used to model rainfall data, with days in Tel Aviv being classified as dry (if no rain falls) and wet otherwise. It gives a reasonable fit to the Tel Aviv rainfall data. For details, see [CM], 3.2.

2. *Gambler's ruin: Random walk with absorbing barriers on a finite set.* Here

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ 0 & q & 0 & p & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & & & \\ \dots & \dots & \dots & \dots & q & 0 & p \\ \dots & \dots & \dots & \dots & 0 & 0 & 1 \end{pmatrix}.$$

Random walk is given by an infinite matrix on the integers, with the tri-diagonal structure above (0 diagonal, p in the super-diagonal, q in the sub-diagonal throughout).

3. *Gambling for fun: Random walk with reflecting barriers on a finite set.* If our gamblers are playing for fun rather than for money, they may decide that to avoid the game stopping when a player is ruined, his last stake is

returned to him so that he can continue playing. The matrix is replaced by

$$P = \begin{pmatrix} q & p & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ 0 & q & 0 & p & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & & & \\ \dots & \dots & \dots & \dots & q & 0 & p \\ \dots & \dots & \dots & \dots & 0 & q & p \end{pmatrix}.$$

4. *Cyclic random walk*. Suppose the states represent positions on a circle:

$$P = \begin{pmatrix} q_0 & q_1 & \dots & \dots & q_{a-1} \\ q_{a-1} & q_0 & \dots & \dots & q_{a-2} \\ \ddots & \ddots & \ddots & \ddots & \\ q_1 & q_2 & \dots & q_{a-1} & q_0 \end{pmatrix}.$$

5. *Ehrenfest model of diffusion: Ehrenfest urn*. Suppose that N balls are distributed between two urns. At each stage, a ball is chosen at random (each with probability $1/N$) and changed to the *other* urn. The state is the number of balls in Urn 1. Then

$$p_{i,i-1} = i/N, \qquad p_{i,i+1} = 1 - i/N, \qquad p_{i,j} = 0 \quad \text{otherwise}$$

(the first represents the chance that a ball in Urn 1 is chosen, and changed to Urn 2, the second that a ball in Urn 2 is chosen, with the complementary probability, and changed to Urn 1). The matrix is again tri-diagonal:

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1/N & 0 & 1-1/N & \dots & 0 & 0 & 0 \\ 0 & 2/N & 0 & 1-2/N & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & & & \\ \dots & \dots & \dots & \dots & 1-1/N & 0 & 1/N \\ \dots & \dots & \dots & \dots & 0 & 1 & 0 \end{pmatrix}.$$

The motivation for this model is Statistical Mechanics (Paul EHRENFEST (1880-1933) and Tatyana Ehrenfest, in 1907, published in 1911). The balls represent molecules of a gas (so for a physically observable system, will be present in enormous numbers – recall *Avogradro's number*, c. 6.02×10^{23}, is the number of gas molecules per standard volume under standard conditions).