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Lecture 6. 26.10.2015 (half-hour: Problems)

The next obvious example is the Normal: if $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, and X, Y are independent, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. This is indeed true, and can be proved as above (try it as an exercise!). We do not do so, because the best way to do this is by another route, to which we now turn.

3. MGFs, CFs and PGFs

Moments. For a random variable X, we define its nth *moment* by

$$\mu_n, \text{ or } \mu_{n,X}, := E[X^n].$$

This need not exist! But, in Statistics, usually *all* moments exist, for $n = 1, 2, \ldots$. (Check which moments exist for the 11 examples in II.1 above.)

Generating functions. Given a function f with Maclaurin expansion

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) x^n / n! = \sum_{n=0}^{\infty} a_n x^n,$$

say, the function f on the LHS is called the *generating function (GF)* of the sequence $a := (a_n)_{n=0}^{\infty}$ on the RHS.

The mathematics of such power-series expansions can only be properly understood in the framework of Complex Analysis (see your undergraduate notes, your textbook of choice, or the M2P3 link on my website). But, you first encountered this in the context of Real Analysis when learning Calculus.

Think of the function f as a way to *encode* the infinitely many a_n.

Example: the Bernoulli numbers. The *Bernoulli numbers* B_n are defined by the power-series expansion (see e.g. my website, M3PM16/M4PM16, Problems 5 Q4)

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n t^n / n!$$

(or can alternatively be defined via the function $x \cot\ x$). One has

$$B_1 = -1/2, \quad B_2 = 1/6, \quad B_3 = 0, \quad B_4 = -1/30, \quad B_5 = 0, \quad B_6 = 1/42,$$

and all higher odd $B_n = 0$. These are important in Analytic Number Theory.

MGFs.

Definition. For a random variable X all of whose moments $\mu_n := E[X^n]$ exist, the *moment-generating function (MGF)* of X is, for t real,

$$M(t), \text{ or } \mu_{n,X}(t), := E[e^{tX}].$$

We now substitute $\sum_{n=0}^{\infty} t^n X^n/n!$ for e^{tX}. If we can interchange $E[.]$ and $\sum_{n=0}^{\infty}$, we obtain

$$M(t) = \sum_{n=0}^{\infty} E[X^n]t^n/n!$$

Thus the MGF on the left is the GF of the moment sequence on the right, hence the name. Recall the *radius of convergence* R of a power series. There are three cases:
(i) $R = 0$. In this case the power series converges only at the origin. It is thus useless. This case can arise – it is possible for all moments to exist but the MGF to have $R = 0$ – but this case does not arise in practice and we exclude it in what follows.
(ii) $0 < R < \infty$. In this case the MGF can be used for some but not all t. In the language of Complex Analysis, $M(t)$ is analytic in $|t| < R$ (for t complex), but there will typically be a singularity on $|t| = R$.
(iii) $R = \infty$. In this case $M(t)$ is entire (= integral – analytic in the whole complex t-plane).
MGFs and convolution.

Recall the Multiplication Theorem: if X, Y are random variables with means (i.e., $E[|X|] < \infty, E[|Y|] < \infty$), then if X, Y are independent,

$$E[XY] = E[X].E[Y].$$

Now if X, Y are independent with MGFs, $X + Y$ has MGF

$$\begin{aligned} M_{X+Y}(t) &:= E[e^{t(X+Y)}] \\ &= E[e^{tX}.e^{tY}] \quad \text{(property of exponentials)} \\ &= E[e^{tX}].E[e^{tY}] \quad (e^{tX}, e^{tY} \text{ are independent as } X, Y \text{ are + Multiplication Th.)} \\ &= M_X(t).M_Y(t) : \end{aligned}$$

the MGF of an independent sum is the product of the MGFs.
Examples.
1. $\chi^2(n)$. If $X_1, X_2, \ldots$ are iid $N(0, 1)$, $\chi^2(n)$ is the distribution of $X_1 + \ldots + X_n^2$. So by above its MGF is the nth power of the MGF of $\chi^2(1)$. This (see Problems, or [BF] Th. 2.1) is $1/(1 - 2t)^{\frac{1}{2}}$ for $t < \frac{1}{2}$. So $\chi^2(n)$ has MGF $1/(1 - 2t)^{\frac{1}{2}n}$. Thus in this case we have $R = \frac{1}{2}$.