pfsl9(15).tex
Lecture 9. 2.11.2015 (half-hour – Problems)

Convergence in probability ('intermediate') implies convergence in distribution ('weak'). We quote this.

There is no converse, but there is a partial converse [which we shall use below]. If X_n converges in distribution to a *constant* c, then since the distribution function of the constant c is 0 to the left of c and 1 at c and to the right, it is easy to see that in fact $X_n \to c$ in probability.

Example. We show by example that convergence in pr does not imply a.s. convergence (a fact known to F. Riesz in 1912). On the *Lebesgue measure space* $[0,1]$ (i.e., $([0,1], \Lambda, \lambda)$, let

$$s_n := 1/2 + 1/3 + \ldots + 1/n \quad (mod\ 1), \quad A_n := [s_{n-1}, s_n], \quad X_n := I_{A_n}.$$

Since the harmonic series diverges, the X_n endlessly move rightwards through the interval $[0,1]$, exiting right and reappearing left. So the X_n do not converge anywhere, in particular are not a.s. convergent. But since $X_n = 0$ except on a set of probability $1/n$, $X_n \to 0$ in probability.

Three classical convergence theorems. We quote (see e.g. SP L6, 8):

M (Lebesgue's monotone convergence theorem). If $X_n \geq 0$, $X_n \uparrow X$, then $E[X_n] \uparrow E[X]$.

F (Fatou's lemma). If $X_n \geq 0$, then $E[\liminf X_n] \leq \liminf E[X_n]$.

D (Lebesgue's dominated convergence theorem). If $X_n \to X$ a.s., $|X_n| \leq Y$ with $E[Y] < \infty$, then $E[X_n] \to E[Y]$.

2. The Weak Law of Large Numbers (WLLN) and the Central Limit Theorem (CLT).

Recall that by Real Analysis,

$$(1 + \frac{x}{n})^n \to e^x \qquad (n \to \infty)$$

(this expresses compound interest, or exponential growth, as the limit of simple interest as the interest is compounded more and more often). This extends also to complex number z, and to $z_n \to z$:

$$(1 + \frac{z_n}{n})^n \to e^z \qquad (n \to \infty).$$

The next result is due to Lévy in 1925, and in more general form to the Russian probabilist A. Ya. KHINCHIN (1894-1956) in 1929 and to Kolmogorov in 1928/29.

Theorem (Weak Law of Large Numbers, WLLN). If X_i are iid with mean μ,

$$\frac{1}{n}\sum_{1}^{n} X_k \to \mu \qquad (n \to \infty) \qquad \text{in probability.}$$

Proof. If the X_k have CF $\phi(t)$, then as the mean μ exists $\phi(t) = 1 + i\mu t + o(t)$ as $t \to 0$. So $(X_1 + \ldots + X_n)/n$ has CF

$$E\exp\{it(X_1 + \ldots + X_n)/n\} = [\phi(t/n)]^n = [1 + \frac{i\mu t}{n} + o(1/n)]^n,$$

for fixed t and $n \to \infty$. By above, the RHS has limit $e^{i\mu t}$ as $n \to \infty$. But $e^{i\mu t}$ is the CF of the constant μ. So by Lévy's continuity theorem,

$$(X_1 + \ldots + X_n)/n \to \mu \qquad (n \to \infty) \qquad \text{in distribution.}$$

Since the limit μ is constant, by II.4 (L11), this gives

$$(X_1 + \ldots + X_n)/n \to \mu \qquad (n \to \infty) \qquad \text{in probability. //}$$

As the name implies, the Weak LLN can be strengthened, to the Strong LLN (with a.s. convergence in place of convergence in probability). We turn to this later, but proceed with a refinement of the method above, in which we retain one more term in the Taylor expansion of the CF. Recall first that the CF of the standard normal distribution $\Phi = N(0,1)$, with density $\phi(x)$ and distribution function $\Phi(x)$

$$\phi(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \qquad \Phi(x) := \int_{\infty}^{x} \phi(u)du$$

is $e^{-t^2/2}$.

Theorem (Central Limit Theorem, CLT). If $X_1, \ldots X_n, \ldots$ are iid with mean μ and variance σ^2, and $S_n := X_1 + \ldots + X_n$, then

$$(S_n - n\mu)/(\sigma\sqrt{n}) \to \Phi = N(0,1) \qquad (n \to \infty) \qquad \text{in distribution.}$$