pfsprob4.tex

PROBLEMS 4. 9.11.2015

Q1. *Means, variances etc. under affine transformations.* Show that:
(i) Means: $E[aX + b] = aEX + b$;
(ii) Variances: $var[aX + b] = a^2\ var[X]$;
(iii) SDs: $SD[aX + b] = |a|SD[X]$;
(iv) Covariances: $cov[aX + b, aY + b] = a^2\ cov[X, Y]$;
(v) Correlations: $corr[aX + b, aY + b] = corr[X, Y]$.
State and prove the corresponding results for sample means etc.

Q2. *Central moments.* Write $\mu_k := E[X^k]$, $\mu_{k,0} := E[(X - EX)^k]$ for the kth moment and central moment of a random variable X. Define the sample central moments, and show that they converge to the (population) central moments (as the sample size n increases).

Q3. *Cumulants.* The *moment-generating function* (MGF) and *cumulant-generating function* (CGF) are

$$M(t) := E[e^{tX}] = \sum_0^\infty \mu_k t^k/k!, \qquad K(t) := \log\ M(t) = \sum_1^\infty \kappa_k t^k/k! = \kappa_1 t + \dots$$

Their centred versions have X replaced by $X - EX$, μ_k, κ_k by $\mu_{k,0}$, $\kappa_{k,0}$. Show that (i) $\kappa_1 = \mu$, $\kappa_k = \kappa_{k,0}$ for $k = 2, 3, \dots$, (ii) $\kappa_2 = \sigma^2$, (iii) $\kappa_3 = \mu_{3,0}$, (iv) $\kappa_{4,0} = \mu_{4,0} - 3\sigma^4$. Show also that X is normal iff all cumulants above the second vanish.

Q4. *Skewness and kurtosis.* Define the *skewness* γ_1 and the *kurtosis* γ_2 by

$$\gamma_1 := \kappa_3/\kappa_2^{3/2} = \mu_{3,0}/\sigma^3, \qquad \gamma_2 := \kappa_4/\kappa_2^2 = \frac{\mu_{4,0}}{\sigma^4} - 3.$$

Show that (i) $\gamma_1[aX+b] = \gamma_1[X]$, $\gamma_2[aX+b] = \gamma_2[X]$; (ii) the sample central moments satisfy $\hat{\mu_{k,0}} \to \mu_{k,0}$ as the sample size n increases; (iii) the sample skewness and kurtosis converge: $\hat{\gamma_1} \to \gamma_1$, $\hat{\gamma_2} \to \gamma_2$.
Note. Skewness measures asymmetry, kurtosis measures thickness of tails, both crucially important for financial data; we can estimate them by (iii).

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