

PROBLEMS 8 7.12.2015

Q1. *First-passage time process of Brownian motion.* For $t \geq 0$ and B BM, write

$$\tau_t := \min\{u : B_u \geq t\}$$

(or $\min\{u : B_u = t\}$ as BM is continuous). So $B(\tau_t) = t$.

(i) Show that for fixed $s \geq 0$, $M_t := \exp\{sB_t - \frac{1}{2}ts^2\}$ is a mg.

(ii) By considering the first-passage times of BM to levels t and $t + u$, show that the process $\tau = (\tau_t)$ is a non-decreasing Lévy process (a subordinator).

(iii) By considering the bounded stopping times $T_n := \min(n, \tau_t)$ and Doob's STP in continuous time and letting $n \rightarrow \infty$, or otherwise, show that

$$E \exp\{-s\tau_t\} = e^{-t\sqrt{2s}}.$$

(iv) Show that for $c > 0$,

$$\tau_t =_d \tau_{ct}/c^2$$

(so τ is *stable* of index $1/2$ – the stable subordinator of index $1/2$).

Q2. Show that τ_1 has density

$$f(x) := \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x^{3/2}} \cdot \exp\left(-\frac{1}{2x}\right).$$

[Let f have Laplace transform $\phi(s) := \int_0^\infty e^{-sx} f(x) dx$. Find $\phi'(s)$, and show, by the change of variable

$$x = \frac{1}{2su}, \quad \text{so} \quad sx = \frac{1}{2u}, \quad \frac{1}{2x} = su,$$

that ϕ satisfies the ODE $\phi'(s)/\phi(s) = -1/\sqrt{2s}$.]

Q3. If X, X_1, \dots, X_n are independent with the stable-1/2 density in Q2, show that $(X_1 + \dots + X_n)/n^2 =_d X$. Why does this not contradict the SLLN?

Q4 (*Scheffé's lemma*). If f_n, f are probability densities, and $f_n \rightarrow f$ a.e., show that (for Borel sets $B \in \mathcal{B}$)

$$\sup_{B \in \mathcal{B}} \left| \int_B f_n - \int_B f \right| \leq \int |f_n - f| \rightarrow 0. \quad \text{NHB}$$