PROBABILITY FOR STATISTICS: RESIT EXAMINATION SOLUTIONS 2014

Q1. The *chi-square distribution* with n *degrees of freedom*, $\chi^2(n)$, is defined as the distribution of $X_1^2 + \ldots + X_n^2$, where X_i are iid $N(0,1)$.
(i) For X standard normal, the MGF of its square X^2 is

$$M(t) := \int e^{tx^2}\phi(x)dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{tx^2}.e^{-\frac{1}{2}x^2}dx.$$

We see that the integral converges only for $t < \frac{1}{2}$, when (normal integral) it is $1/\sqrt{(1-2t)}$: $\quad M(t) = 1/\sqrt{(1-2t)}$ $(t < \frac{1}{2})$ for $X \sim N(0,1)$.
So by definition of $\chi^2(n)$, the MGF of a $\chi^2(n)$ is

$$M(t) = 1/(1-2t)^{\frac{1}{2}n} \qquad (t < \frac{1}{2}) \qquad \text{for } X \sim \chi^2(n). \qquad [\mathbf{8}]$$

Replacing t by it by analytic continuation, the characteristic function is

$$\phi(t) = 1/(1-2it)^{\frac{1}{2}n}.$$

(ii) First, the required density $f(.)$ is a density, as $f \geq 0$ and $\int f = 1$:

$$\int f(x)dx = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x)dx = \frac{1}{\Gamma(\frac{1}{2}n)}.\int_0^\infty u^{\frac{1}{2}n-1}\exp(-u)du = 1$$

$(u := \frac{1}{2}x)$, by definition of the Gamma function. Its MGF is

$$M(t) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty e^{tx}.x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x)dx = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}.\int_0^\infty x^{\frac{1}{2}n-1}\exp(-\frac{1}{2}x(1-2t))dx.$$

Substitute $u := \frac{1}{2}x(1-2t)$ in the integral. One obtains

$$M(t) = (1-2t)^{-\frac{1}{2}n}.\frac{1}{\Gamma(\frac{1}{2}n)}.\int_0^\infty u^{\frac{1}{2}n-1}e^{-u}du = (1-2t)^{-\frac{1}{2}n}.$$

So it has the required MGF, so is the required density. // **[9]**
(iii) For $n = 1$, the mean is 1, because a $\chi^2(1)$ is the square of a standard normal, and a standard normal has mean 0 and variance 1. The variance is 2, because the fourth moment of a standard normal X is 3, and

$$var(X^2) = E[(X^2)^2] - [E(X^2)]^2 = 3 - 1 = 2.$$

For general n, the mean is n because means add. **[4]**
The variance is $2n$ because variances add over independent summands.
(Or, use (i), (ii).) **[4]**
[Seen – problems + Mock exam]

Q2 (*Weak Law of Large Numbers, WLLN*).

Random variables X_n *converge in probability* to X if

$$\forall\, \epsilon > 0, \quad P(|X_n - X| > \epsilon) \to 0 \qquad (n \to \infty). \qquad \textbf{[2]}$$

They converge to X *in distribution* if

$$F_n(x) := P(X_n \le x) \to F(x) := P(X \le x) \qquad (n \to \infty),$$

at all continuity points x of F. **[2]**

(a) Convergence in probability implies convergence in distribution, but not conversely in general. **[1]**

(b) The converse holds (so the two are equivalent) if the limit X is constant. **[1]**

We need the following properties of the characteristic function (CF):

(i) (Lévy's convergence theorem). Convergence in distribution of random variables is equivalent to convergence of CFs (uniformly on compacta). **[2]**

(ii). The CF of an independent sum is the product of the CFs. **[2]**

(iii). If the random variable has k moments (finite), the CF can be expanded as far as the t^k term with negligible error term $o(t^k)$ for small t. **[2]**

Recall that (for x real and $z_n \to z$ complex)

$$(1 + \frac{x}{n})^n \to e^x, \qquad (1 + \frac{z_n}{n})^n \to e^z \qquad (n \to \infty). \qquad (*) \ \textbf{[2]}$$

Theorem (Weak Law of Large Numbers, WLLN). **[3]**

If X_i are iid with mean μ,

$$\frac{1}{n}\sum_1^n X_k \to \mu \qquad (n \to \infty) \qquad \text{in probability.}$$

Proof. If the X_k have CF $\phi(t)$, then as the mean μ exists $\phi(t) = 1 + i\mu t + o(t)$ as $t \to 0$, by (iii). So by (ii) $(X_1 + \ldots + X_n)/n$ has CF

$$E\exp\{it(X_1 + \ldots + X_n)/n\} = [\phi(t/n)]^n = [1 + \frac{i\mu t}{n} + o(1/n)]^n,$$

for fixed t and $n \to \infty$. By $(*)$, the RHS has limit $e^{i\mu t}$ as $n \to \infty$. But $e^{i\mu t}$ is the CF of the constant μ. So by Lévy's continuity theorem (i),

$$(X_1 + \ldots + X_n)/n \to \mu \qquad (n \to \infty) \qquad \text{in distribution.}$$

Since the limit μ is constant, this and (b) give

$$(X_1 + \ldots + X_n)/n \to \mu \qquad (n \to \infty) \qquad \text{in probability. //} \qquad \textbf{[8]}$$

[Seen – lectures + Exam 2014].

Q3 (*Compound Poisson processes*).

Let jumps $\{X_1, \cdots, X_n, \cdots\}$ arrive at the epochs of a Poisson process with rate λ [claims, in the insurance context used in lectures – call them claims below, for definiteness]. Then the number $N(t)$ of claims in the time-interval $[0, t]$ is Poisson $P(\lambda t)$. If the claims are iid with mean μ and CF $\phi(u)$, then the claim total at time t is $S(t) := X_1 + \cdots X_{N(t)}$, with CF

$$\psi(u) := E[\exp\{iuS(t)\}] = E[\exp\{iu(X_1 + \cdots + X_{N(t)})]$$

$$= \sum_{n=0}^{\infty} E[\exp\{iu(X_1 + \cdots + X_{N(t)})\}|N(t) = n]P(N(t) = n)$$

$$= \sum E[\exp\{iu(X_1 + \cdots + X_n)\}].e^{-\lambda t}(\lambda t)^n/n! = e^{-\lambda t}\exp\{\lambda t\phi(u)\}. \quad [\mathbf{10}]$$

From

$$\phi(u) := E[e^{iuX}], \qquad \phi'(u) = iE[Xe^{iuX}], \qquad \phi''(u) = -E[X^2e^{iuX}],$$

so

$$\phi'(0) = iE[X] = i\mu,$$

say,

$$\phi''(0) = -E[X^2],$$

and similarly

$$\psi'(0) = iE[S(t)], \qquad \psi''(0) = -E[S(t)^2]. \quad [\mathbf{3}]$$

So differentiating,

$$\psi'(u) = \phi'(u).\lambda t.\psi(u); \qquad \psi'(0) = \lambda t.\phi'(0);$$

$$\psi''(u) = \lambda t.\phi''(u).\psi(u) + \lambda t.\phi'(u), \psi'(u)$$

$$= \lambda t\phi''(u)\psi(u) + (\lambda t)^2[\phi'(u)]^2\psi(u) :$$

$$\psi''(0) = \lambda t\phi''(0) + (\lambda t)^2[\phi'(0)]^2. \quad [\mathbf{3}]$$

Combining,

$$E[S(t)] = \lambda tE[X] = \lambda t\mu; \quad [\mathbf{4}]$$

$$var[S(t)] = -\psi''(0) = [\psi'(0)]^2$$

$$= -\lambda t\phi''(0) + (\lambda t)^2\mu^2 - (\lambda t)^2\mu^2$$

$$= \lambda tE[X^2]. \quad [\mathbf{5}]$$

[Seen – problems + Exam 2014]

Q4 (*Finite Markov chains*).

A state i in a Markov chain is *transient* if the chain spends only finitely long in i (a.s.), *recurrent* (or *persistent*) otherwise. **[1,1]**

The *mean recurrence time* of a state i is the expectation of the time T_i of first return to i, starting at i. **[1]**

A recurrent state (one to which the chain returns infinitely often (i.o.), a.s.) is *positive* if the mean recurrence time is finite, *null* otherwise. **[1,1]**

Theorem. For a finite Markov chain, it is impossible for all states to be transient: a finite chain must contain at least one persistent state.

Proof. If the state-space is $\{1, \cdots, N\}$, for each i and each n

$$1 = \sum_{j=1}^{N} p_{ij}(n). \qquad (a)$$

Let $n \to \infty$: if j is transient, the total expected time in it is finite: $\sum_n p_{ij}(n) < \infty$. So

$$p_{ij}(n) \to 0 \qquad (n \to \infty). \qquad (b)$$

Were *all* states transient, letting $n \to \infty$ in (a) and using (b) would give the contradiction $1 = 0$. So not all states in a finite chain can be transient. // **[7]**

Theorem. A recurrent state j in a finite chain is positive (= non-null).

Proof. If the finite chain has state-space $\{1, \cdots, N\}$, assume there is a null state. Let C be the equivalence class containing it. Since C is closed, we can consider the subchain induced on C. Then

$$1 = \sum_{k \in C} p_{ik}(n) \qquad \text{(finite sum)}.$$

Let $n \to \infty$: each $p_{ik}(n) \to 0$, so RHS $\to 0$, giving $1 = 0$. This contradiction gives the non-existence of null states in a finite chain. // **[7]**

All states may be :

(i) transient. Trivial example: $\mathbb{Z}$, moving to the right at each step. **[2]**

(ii) positive recurrent. Trivial example: $\mathbb{Z}$, with each state a trap. **[2]**

(iii) null recurrent. Example: simple random walk on $\mathbb{Z}$. **[2]**

[Seen – lectures].

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