

## SOLUTIONS 2. 2.11.2015

Q1. (i) The joint density of the  $x_i$  is

$$f(x) = (2\pi)^{-\frac{1}{2}n} \prod_{i=1}^n \exp\left\{-\frac{1}{2}x_i^2\right\} = (2\pi)^{-\frac{1}{2}n} \exp\left\{-\frac{1}{2}\sum_1^n x_i^2\right\} = (2\pi)^{-\frac{1}{2}n} \exp\left\{-\frac{1}{2}\|x\|^2\right\}.$$

The Jacobian of the change of variable is the determinant  $|O|$ ,  $= 1$  as  $O$  is orthogonal (= length-preserving). So the joint density of the  $y_i$  is

$$g(y) = (2\pi)^{-\frac{1}{2}n} \exp\{-\|y\|^2\} = (2\pi)^{-\frac{1}{2}n} \exp\left\{-\sum_1^n y_i^2\right\},$$

which says that the  $y_i$  are iid  $N(0, 1)$ .

(ii) The condition for a matrix  $O$  to be orthogonal is that the rows are of length 1 and orthogonal vectors. Take the first row as  $e_1$ , and use Gram-Schmidt orthogonalisation to find  $e_2$  orthogonal to  $e_1$ , then  $e_3$  orthogonal to  $e_1, e_2$  etc. The  $e_i$  form the rows of an orthogonal matrix with first row  $e_1$ .

(iii) Put  $Z_i := (X_i - \mu)/\sigma$ ,  $Z := (Z_1, \dots, Z_n)^T$ ; then the  $Z_i$  are iid  $N(0, 1)$ ,

$$\bar{Z} = (\bar{X} - \mu)/\sigma, \quad nS^2/\sigma^2 = \sum_1^n (Z_i - \bar{Z})^2.$$

Also

$$\sum_1^n Z_i^2 = \sum_1^n (Z_i - \bar{Z})^2 + n\bar{Z}^2,$$

since  $\sum_1^n Z_i = n\bar{Z}$ . The terms on the right above are quadratic forms, with matrices  $A$ ,  $B$  say, so we can write

$$\sum_1^n Z_i^2 = Z^T A Z + Z^T B Z. \quad (*)$$

Put  $W := PZ$  with  $P$  a Helmert transformation with first row  $(1, \dots, 1)/\sqrt{n}$ :

$$W_1 = \frac{1}{\sqrt{n}} \sum_1^n Z_i = \sqrt{n}\bar{Z}; \quad W_1^2 = n\bar{Z}^2 = Z^T B Z.$$

So by above,

$$Z^T A Z = \sum_1^n (Z_i - \bar{Z})^2 = nS^2/\sigma^2 = \sum_2^n W_i^2,$$

as  $\sum_1^n Z_i^2 = \sum_1^n W_i^2$ . But the  $W_i$  are independent (by the orthogonality of  $P$ ), so  $W_1$  is independent of  $W_2, \dots, W_n$ . So  $W_1^2$  is independent of  $\sum_2^n W_i^2$ . So  $nS^2/\sigma^2$  is independent of  $n(\bar{X} - \mu)^2/\sigma^2$ , so  $S^2$  is independent of  $\bar{X}$ , as claimed.

Q2. (i) For  $n = 1$ , the mean is 1, because a  $\chi^2(1)$  is the square of a standard normal, and a standard normal has mean 0 and variance 1. The variance is 2, because the fourth moment of a standard normal  $X$  is 3, and

$$\text{var}(X^2) = E[(X^2)^2] - [E(X^2)]^2 = 3 - 1 = 2.$$

For general  $n$ , the mean is  $n$  because means add, and the variance is  $2n$  because variances add over independent summands.

(ii) For  $X$  standard normal, the MGF of its square  $X^2$  is

$$M(t) := \int e^{tx^2} \phi(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{tx^2} \cdot e^{-\frac{1}{2}x^2} dx.$$

We see that the integral converges only for  $t < \frac{1}{2}$ , when it is  $1/\sqrt{(1-2t)}$ :

$$M(t) = 1/\sqrt{1-2t} \quad (t < \frac{1}{2}) \quad \text{for } X \sim N(0, 1).$$

So by definition of  $\chi^2(n)$ , the MGF of a  $\chi^2(n)$  is

$$M(t) = 1/(1-2t)^{\frac{1}{2}n} \quad (t < \frac{1}{2}) \quad \text{for } X \sim \chi^2(n).$$

(iii) First,  $f(\cdot)$  is a density, as it is non-negative, and integrates to 1:

$$\begin{aligned} \int f(x) dx &= \frac{1}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)} \cdot \int_0^\infty x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x) dx \\ &= \frac{1}{\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty u^{\frac{1}{2}n-1} \exp(-u) du \quad (u := \frac{1}{2}x) \\ &= 1, \end{aligned}$$

by definition of the Gamma function. Its MGF is

$$M(t) = \frac{1}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)} \cdot \int_0^\infty e^{tx} \cdot x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x) dx = \frac{1}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)} \cdot \int_0^\infty x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x(1-2t)) dx.$$

Substitute  $u := \frac{1}{2}x(1-2t)$  in the integral. One obtains

$$M(t) = (1-2t)^{-\frac{1}{2}n} \cdot \frac{1}{\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty u^{\frac{1}{2}n-1} e^{-u} du = (1-2t)^{-\frac{1}{2}n},$$

by definition of the Gamma function. //

NHB