pfssoln3.tex

SOLUTIONS 3. 9.11.2015

Q1.

$$\Gamma(x+a) \sim \sqrt{2\pi} e^{-x-a} (x+a)^{x+a-\frac{1}{2}} \sim \sqrt{2\pi} e^{-x} x^{x+a-\frac{1}{2}} \Big(1+\frac{a}{x}\Big)^{x+a-\frac{1}{2}}$$

$$\sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} . x^a . e^{-a} \Big(1+\frac{a}{x}\Big)^x \sim \Gamma(x).x^a,$$

using Stirling's formula and $(1+a/x)^x \to e^a$.

Q2. (i) We show that for $t(r)$, the density $f(x) \to \phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ as $r \to \infty$. As $(1+x/n)^n \to e^x$ as $n \to \infty$ ('compound interest'), the bracket tends to $e^{-\frac{1}{2}x^2}$.

By Q1,

$$\Gamma(r+a) \sim r^a \Gamma(r) \qquad (r \to \infty).$$

So the ratio of Γs $\sim \frac{1}{2} r^{\frac{1}{2}}$. So the constant $\sim 1/\sqrt{2\pi}$. Combining, $f(x) \to \phi(x)$, as required.

(ii) $\bar{X} \sim N(\mu, \sigma^2/n)$, so $\sqrt{n}(\bar{X}-\mu)/\sigma \sim N(0,1)$. But

$$S^2 \to \sigma^2, \qquad S \to \sigma \qquad (n \to \infty),$$

by the Law of Large Numbers ('Law of Averages' – L10), and $\sqrt{n-1} \sim \sqrt{n}$. Combining,

$$t(n-1) \to N(0,1).$$

Q3.

$$\begin{aligned} Q(\lambda) &= \lambda \frac{1}{n} \sum_1^n (x_i - \bar{x})^2 + 2\lambda \frac{1}{n} \sum_1^n (x_i - \bar{x})(y_i - \bar{y}) + \frac{1}{n} \sum_1^n (y_i - \bar{y})^2 \\ &= \lambda^2 \overline{(x-\bar{x})^2} + 2\lambda \overline{(x-\bar{x})(y-\bar{y})} + \overline{(y-\bar{y})^2} \\ &= \lambda^2 S_{xx} + 2\lambda S_{xy} + S_{yy}. \end{aligned}$$

Now $Q(\lambda) \geq 0$ for all λ, so $Q(\cdot)$ is a quadratic which does not change sign. So its discriminant is ≤ 0 (if it were > 0, there would be distinct real roots and a sign change). So ("$b^2 - 4ac \leq 0$")

$$s_{xy}^2 \leq s_{xx} s_{yy} = s_x^2 s_y^2, \qquad r^2 := (s_{xy}/s_x s_y)^2 \leq 1.$$

So

$$-1 \leq r \leq +1.$$

The extremal cases $r = \pm 1$ or $r^2 = 1$, have discriminant 0, that is $Q(\lambda)$ has a repeated real root, λ_0 say. But then $Q(\lambda_0)$ is the sum of squares of $\lambda_0(x_i - \bar{x}) + (y_i - \bar{y})$, which is zero. So each term is 0:

$$\lambda_0(x_i - \bar{x}) + (y_i - \bar{y}) = 0 \quad (i = 1 \ldots n).$$

That is, all the points (x_i, y_i) $(i = 1 \ldots n)$, lie on a straight line through the centroid $(\bar{x}, \bar{y})$ with slope $-\lambda_0$.

Q4. Similarly

$$\begin{aligned} Q(\lambda) &= E[\lambda^2(x - Ex)^2 + 2\lambda(x - Ex)(y - Ey) + (y - Ey)^2] \\ &= \lambda^2 E[(x - Ex)^2] + 2\lambda E[(x - Ex)(y - Ey)] + E[(y - Ey)^2] \\ &= \lambda^2\sigma_x^2 + 2\lambda\sigma_{xy} + \sigma_y^2. \end{aligned}$$

As before $Q(\lambda) \geq 0$ for all λ, as the discriminant is ≤ 0, i.e.

$$\sigma_{xy}^2 \leq \sigma_x^2\sigma_y^2, \quad \rho := (\sigma_{xy}/\sigma_x\sigma_y)^2 \leq 1, \quad -1 \leq \rho \leq +1.$$

The extreme cases $\rho = \pm 1$ occur iff $Q(\lambda)$ has a repeated real root λ_0. Then

$$Q(\lambda_0) = E[(\lambda_0(x - Ex) + (y - Ey))^2] = 0.$$

So the random variable $\lambda_0(x - Ex) + (y - Ey)$ is zero (a.s. – except possibly on some set of probability 0). So all values of (x, y) lie on a straight line through the centroid (Ex, Ey) of slope $-\lambda_0$, a.s.

Note. A slight extension of this argument, using an inner product on a complex vector space, works with complex numbers and leads to conclusions of the form $|r| \leq 1$, $|\rho| \leq 1$. For details, see e.g. Section 2.3 of
D. J. H. GARLING, *Inequalities: A journey into linear analysis*, CUP, 2007.
A real inner product is *bilinear*. A complex inner product is *sesquilinear*: *linear* in the first argument,

$$\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle,$$

but *antilinear* in the second argument:

$$\langle f, ag + bh\rangle = \bar{a}\langle f, g\rangle + \bar{b}\langle f, h\rangle.$$

In this course, we will deal mainly with *real* inner products and Hilbert space, but the *complex* case is very important. NHB