pfssoln7.tex

SOLUTIONS 7 7.12.2015

Q1: *Hypergeometric distribution.* (i) The number of subsets of size n of a set of size $2n$ is $\binom{2n}{n}$. If this subset contains k white balls, these can be chosen in $\binom{n}{k}$ ways; the remaining $n-k$ balls are black, and can be chosen in $\binom{n}{n-k} = \binom{n}{k}$ ways, giving $\binom{n}{k}^2$ ways altogether; sum over k.
(ii)

$$\sum_i \binom{2n}{i} x^i = (\sum_j \binom{n}{j} x^j)(\sum_k \binom{n}{k} x^k).$$

Extracting the coefficient of x^n gives $\binom{2n}{n}$ on the left and $\sum_j \binom{n}{j}\binom{n}{n-j} = \sum_j \binom{n}{j}^2$ on the right.
(iii) The number of routes from the vertex to the central element in row $2n$ is $\binom{2n}{n}$. There are $\binom{n}{k}$ routes from the vertex to the element $\binom{n}{k}$ in row n. By symmetry of the "square" with top corner the vertex and bottom corner $\binom{2n}{n}$ about its horizontal diagonal, the number of routes from $\binom{n}{k}$ to $\binom{2n}{n}$ is $\binom{n}{k}$. So there are $\binom{n}{k}^2$ routes passing through $\binom{n}{k}$; sum over k.

Q2: *Bernoulli-Laplace urn.* With π the hypergeometric distribution given (this is a probability distribution, by Q1),

$$\pi_i p_{i,i+1} = \frac{1}{\binom{2d}{d}} \binom{d}{i}^2 . \left(\frac{d-i}{d}\right)^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2,$$

and similarly

$$\pi_{i+1} p_{i+1,i} = \frac{1}{\binom{2d}{d}} \binom{d}{i+1}^2 . \left(\frac{i+1}{d}\right)^2 = \frac{1}{\binom{2d}{d}} \binom{d-1}{i}^2,$$

proving detailed balance, and so reversibility. Assuming reversibility, we can use detailed balance to calculate the invariant distribution:

$$\pi_i = \frac{\pi_0}{\left(\frac{1}{d}\right)^2} . \frac{\left(1-\frac{1}{d}\right)^2}{\left(\frac{2}{d}\right)^2} . \ldots . \frac{\left(1-\frac{i-1}{d}\right)^2}{\left(\frac{i}{d}\right)^2} = \pi_0 . \frac{(d(d-1)\ldots(d-i+1))^2}{(1.2\ldots.i)^2} = \pi_0 \binom{d}{i}^2 .$$

Then $\sum_i \pi_i = 1$ gives

$$\pi_0 \sum_i \binom{d}{i}^2 = \pi_0 \binom{2d}{d}^2 = 1, \qquad \pi_0 = 1/\binom{2d}{d}, \qquad \pi_i = \binom{d}{i}^2 / \binom{2d}{d}. \quad //$$

Q3: *Bernoulli-Laplace urn (continued).* $\pi_i = 1/\mu_i$ by the Erdös-Feller-Pollard theorem (L19), so

$$\mu_0 = 1/\pi_0 = \binom{2d}{d}.$$

By Stirling's formula,

$$\mu_0 \sim \frac{\sqrt{2\pi} e^{-2d} 2d^{2d+\frac{1}{2}}}{(\sqrt{2\pi} e^{-d} d^{d+\frac{1}{2}})^2} = \frac{4^d}{\sqrt{\pi d}}.$$

Now as d is already very large (of the order of Avogadro's number 6×10^{23}), 4^d is astronomically vast – effectively infinite.

The interpretation of this in Statistical Mechanics is that μ_0 is the mean recurrence time of state 0, when all the $2d$ gas molecules are in one half of the container. Although this state is certain to recur (indeed, infinitely often), its mean recurrence time is so vast as to be effectively infinite – which explains why we do not see such states recurring in practice! This reconciles the theoretical reversibility of the model with the irreversible behaviour we observe when gases diffuse, etc. This was the Ehrenfests' motivation for their model, in 1912.

Note.

Entropy.

Relevant here is the concept of *entropy* – a measure of disorder. This was introduced by Rudolf CLAUSIUS (1922-1888), in 1865, who formulated the Fist Law of Thermodynamics (Law of Conservation of Energy) and Second Law of Thermodynamics (entropy increases – things become more disordered):

1. Die Energie der Welt ist konstant (The energy of the world [the universe] is constant).

2. Die Entropie der Welt strebt einem Maximum zu (The entropy of the world [the universe] strives towards a maximum).

This is worth learning – it is the most famous two-sentence passage in the history of science.

Boltzmann, the Bernoulli-Laplace urn and the Ehrenfest urn.

The following is a tragic story. The Bernoulli-Laplace urn (BL) originates in the work of Daniel BERNOULLI (1700 - 1782) in 1770, and Pierre-Simone de LAPLACE (1749 - 1827) in 1810-11 and 1812. It is more complicated than its relative, the Ehrenfest urn (E), which was specifically developed to reconcile the irreversible macrodynamics but reversible microdynamics above.

In addition to Clausius and Thermodynamics, the relevant area here is Statistical Mechanics, largely the creation of three people: the German Ludwig BOLTZMANN (1844 - 1906; H-theorem, 1872: entropy increases), the Scotsman James Clerk MAXWELL (1831 - 1916), and the American Josiah Willard GIBBS (1839 - 1903).

Atoms and molecules were not experimentally observable at that time, and their existence was aggressively challenged by some scientists of the period, particularly the German physicist Ernst Mach (1838 - 1916: "I do not believe in atoms", 1897), now remembered mainly for Mach 1, the speed of sound, Mach 2, double it, etc. Had Boltzmann known of BL, he would have been able to use it to refute Mach's constant attacks. Similarly had E been available at that time (though this story is part of the motivation for the development of E). Boltzmann was a fine scientist, but not a great debater; he unfortunately saw it as his duty to take on his critics in debate; his mental health collapsed under the strain; he tragically committed suicide.
References:

David LINDLEY: *Boltzmann's atom: The great debate that launched a revolution in physics*. The Free Press, 2001;
Martin JACOBSEN: Laplace and the origin of the Ornstein-Uhlenbeck process. *Bernoulli* **2**.3 (1996), 271 - 286.

One moral of this sad story is that it is all too easy for work developed in one field, and in one period, to be overlooked by later workers in other areas. Bernoulli and Laplace could hardly have anticipated Statistical Mechanics, or Thermodynamics (then still in its early days – work of Carnot, etc.). Equally, Boltzmann, Maxwell and Gibbs could hardly have known all the probability then available, still less seen how to apply it.

The existence of atoms and molecules was established indirectly, starting with Einstein's work of 1905 on Brownian motion and diffusion (2005 was Einstein Year, in honour of his three pioneering papers – on this, the photo-electric effect (for which he got the Nobel Prize) and special relativity). There was related work by Langevin in 1914, and Smoluchowski in 1918 ('Einstein-Smoluchowski theory).

Q4: *Branching processes.*
(i) Z_2 is the sum of a random number, Z_1, of independent copies of Z. So

$$P_2(s) := E[s^{Z_2}] = \sum_{k=0}^{\infty} E[s^{Z_2}|Z_1 = k]P(Z_1 = k).$$

Now when $Z_1 = k$, Z_2 is a sum of k independent copies of Z, each with PGF $P(s)$, so has (conditional) PGF $P(s)^k$. So

$$P_2(s) = \sum_0^{\infty} p_k P(s)^k = P(P(s)).$$

(ii) Similarly, or by induction on n, Z_n has PGF P_n.
(iii)

$$P_n'(s) = P'(P_{n-1}(s)).P_{n-1}'(s).$$

So letting $s = 1$ ($R > 1$), or $s \uparrow 1$ ($R = 1$) and using Abel's Continuity Theorem, since P_{n-1}, being a PGF, has value 1 at 1, $P_n'(1) = P'(1).P_{n-1}'(1) = \mu.P_{n-1}'(1)$, so by induction

$$P_n'(1) = \mu^n : \quad E[Z_n] = \mu^n.$$

NHB