pfssoln9.tex

SOLUTION 9 15.12.2015

Q1. We have to check the defining property (CE) (V.1, L18) for $B = \emptyset$ and $B = \Omega$. For $B = \emptyset$ both sides are zero; for $B = \Omega$ both sides are EY. //
(ii) We have to check (CE) for *all* sets $B \in \mathcal{A}$. The only integrand that integrates like Y over *all* sets is Y itself, or a function agreeing with Y except on a set of measure zero.
(iii) Recall that Y is *always* $\mathcal{A}$-measurable (this is the definition of Y being a random variable). For $\mathcal{B} \subset \mathcal{A}$, Y may not be $\mathcal{B}$-measurable, but if it is, the proof above applies with $\mathcal{B}$ in place of $\mathcal{A}$.
(iv) (Tower property). If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{B})\ |\mathcal{C}] = E[Y|\mathcal{C}]$ a.s.

Proof. $E_{\mathcal{C}} E_{\mathcal{B}} Y$ is $\mathcal{C}$-measurable, and for $C \in \mathcal{C} \subset \mathcal{B}$,

$$\int_C E_{\mathcal{C}}[E_{\mathcal{B}} Y] dP = \int_C E_{\mathcal{B}} Y dP \qquad (\text{definition of } E_{\mathcal{C}} \text{ as } C \in \mathcal{C})$$

$$= \int_C Y dP \qquad (\text{definition of } E_{\mathcal{B}} \text{ as } C \in \mathcal{B}).$$

So $E_{\mathcal{C}}[E_{\mathcal{B}} Y]$ satisfies the defining relation for $E_{\mathcal{C}} Y$. Being also $\mathcal{C}$-measurable, it *is* $E_{\mathcal{C}} Y$ (a.s.). //
9 (iv') (Tower property). If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{C})\ |\mathcal{B}] = E[Y|\mathcal{C}]$ a.s.
Proof. $E[Y|\mathcal{C}]$ is $\mathcal{C}$-measurable, so $\mathcal{B}$-measurable as $\mathcal{C} \subset \mathcal{B}$, so $E[.|\mathcal{B}]$ has no effect, by (iii). //
Corollary. $E[E(Y|\mathcal{C})\ |\mathcal{C}] = E[Y|\mathcal{C}]$ a.s.
So the operation $E[.|\mathcal{C}]$ is linear and *idempotent* (doing it twice is the same as doing it once), so is a *projection*. So we can use what we know about projections, from Ch. IV, Linear Algebra, Functional Analysis etc.

Q2. (i) For $t \neq 0$, X is Gaussian with zero mean (as B is), and continuous (again, as B is). The covariance of B is $\min(s,t)$. The covariance of X is

$$cov(X_s, X_t) = cov(sB(1/s), tB(1/t)) = E[sB(1/s).tB(1/t)] = st.E[B(1/s)B(1/t)]$$

$$= st.cov(B(1/s), B(1/t)) = st.\min(1/s, 1/t) = \min(t,s) = \min(s,t).$$

This is the same covariance as Brownian motion. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by

its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.
(ii) Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \to 0$ as $t \to 0$. This says that

$$tB(1/t) \to 0 \qquad (t \to 0), \quad \text{i.e.} \qquad B(t)/t \to 0 \qquad (t \to \infty).$$

Note. For t integer, this is the Strong Law of Large Numbers applied to the distribution of $B(1)$, which is standard normal. The above neat proof by *time-inversion* follows from the proof of existence of Brownian motion (defined to be continuous), given in lectures by the PWZ wavelet expansion.

Q3. Brownian bridge $X_t := B_t - tB_1$ ($t \in [0, 1]$) is Gaussian (it is obtained from the Gaussian process B by linear operations – as in the multivariate normal distribution, IV.3). It has mean 0 (as B does), and covariance

$$E[X_s X_t] = E[(B_s - sB_1)(B_t - tB_1)] = E[B_s B_t] - tE[B_s B_1] - sE[B_t B_1] + stE[B_1^2]$$

$$= \min(s, t) - t \min(s, 1) - s \min(t, 1) + st = \min(s, t) - st - st + st = \min(s, t) - st.$$

NHB