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Lecture 8. 4.2.2011

4. GENERAL AUTOREGRESSIVE PROCESSES, AR(p).

Again working with the zero-mean case for simplicity, the extension of the above to p parameters is the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t, \qquad (*)$$

with (ϵ_t) WN as before. Since $X_{t-i} = B^i X_t$, we may re-write this as

$$X_t - \phi_1 B X_t - \cdots - \phi_p B^p X_t = \epsilon_t.$$

Write

$$\phi(\lambda) := 1 - \phi\lambda - \cdots - \phi_p \lambda^p$$

for the pth order polynomial here. Then formally,

$$\phi(B) X_t = \epsilon_t.$$

Formally again,

$$X_t = \phi(B)^{-1} \epsilon_t,$$

so if we expand $1/\phi(\lambda)$ in a power series as

$$1/\phi(\lambda) \equiv 1 + \beta_1 \lambda + \cdots + \beta_n \lambda^n + \cdots,$$

$$X_t = \sum\nolimits_{i=0}^{\infty} \beta_i B^i \epsilon_t = \sum\nolimits_0^{\infty} \beta_i \epsilon_{t-i}.$$

This is the analogue of $X_t = \sum_0^\infty \phi^i \epsilon_{t-i}$ for $AR(1)$, and shows that X_t can again be represented as an infinite moving-average process - or *linear process* (X_t is an (infinite) *linear combination* of the ϵ_{t-i}).

Multiply $(*)$ through by X_{t-k} and take expectations. Since $E[X_{t-k}X_{t-i}] = \rho(|k-i|) = \rho(k-i)$, this gives

$$\rho(k) = \phi_1 \rho(k-1) + \cdots + \phi_p \rho(k-p) \qquad (k > 0). \qquad (YW)$$

These are the *Yule-Walker equations*, due to G. Udny Yule (1871-1951) in 1926 and Sir Gilbert Walker (1868-1958) in193.

The Yule-Walker equations (YW) have the form of a *difference equation* of *order p*. The *characteristic polynomial* of this difference equation is

$$\lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_p = 0,$$

which by above is

$$\phi(1/\lambda) = 0.$$

If $\lambda_1, \cdots, \lambda_p$ are the roots of this characteristic polynomial, the trial solution $\rho(k) = \lambda^k$ is a solution if and only if λ is one of the roots λ_i. Since the equation is linear,

$$\rho(k) = c_1\lambda_1^k + \cdots + c_p\lambda_p^k$$

(for $k \geq 0$, and use $\rho(-k) = \rho(k)$ for $k < 0$) is a solution for all choices of constants $c_1, \cdots, c_p$. This is the *general solution* of (YW) if all the roots λ_i are distinct, with appropriate modifications for repeated roots (if $\lambda_1 = \lambda_2$, use $c_1\lambda_1^k + c_2 k\lambda_1^k$, etc.).

Now $|\rho(k)| \leq 1$ for all k (as $\rho(.)$ is a correlation coefficient), and this is *only possible* if

$$|\lambda_i| \leq 1 \qquad (i = 1, \cdots, p)$$

- that is, all the roots lie inside (or on) the unit circle. This happens (as our polynomial is $\phi(1/\lambda)$) if and only if *all the roots of the polynomial* $\phi(\lambda)$ *lie outside (or on) the unit circle.* Then $|\rho(k)| \leq 1$ for all k, and when there are no roots of unit modulus, also $\rho(k) \to 0$ as $k \to \infty$ - that is, the influence of the remote past tends to zero, as it should. We shall see below that this is also the condition for the $AR(p)$ process above to be *stationary.*

Example of an $AR(2)$ *process.*

$$X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + \epsilon_t, \qquad (\epsilon_t) \quad WN. \tag{1}$$

Moving-average representation. Let the infinite moving-average representation of (X_t) be

$$X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}. \tag{2}$$

Substitute (2) into (1):

$$\begin{aligned} \sum_0^{\infty} \psi_i \epsilon_{t-i} &= \frac{1}{3}\sum_0^{\infty} \psi_i \epsilon_{t-i-1} + \frac{2}{9}\sum_0^{\infty} \psi_i \epsilon_{t-2-i} + \epsilon_t \\ &= \frac{1}{3}\sum_1^{\infty} \psi_{i-1} \epsilon_{t-i} + \frac{2}{9}\sum_2^{\infty} \psi_{i-2} \epsilon_{t-i} + \epsilon_t. \end{aligned}$$

Equate coefficients of ϵ_{t-i}:
$i = 0$ gives $\psi_0 = 1$; $i = 1$ gives $\psi_1 = \frac{1}{3}\psi_0 = 1/3$; $i \geq 2$ gives

$$\psi_i = \frac{1}{3}\psi_{i-1} + \frac{2}{9}\psi_{i-2}.$$

This is again a difference equation, which we solve as above. The characteristic polynomial is

$$\lambda^2 - \frac{1}{3}\lambda - \frac{2}{9} = 0, \qquad \text{or} \qquad (\lambda - \frac{2}{3})(\lambda + \frac{1}{3}) = 0,$$

with roots $\lambda_1 = 2/3$ and $\lambda_2 = -l/3$. The general solution of the difference equation is thus $\psi_i = c_1\lambda_1^i + c_2\lambda_2^i = c_1(2/3)^i + c_2(-1/3)^i$. We can find c_1, c_2 from the values of ψ_0, ψ_1, found above:
$i = 0$ gives $c_1 + c_2 = 0$, or $c_2 = 1 - c_1$.
$i = 1$ gives $c_1.(2/3)+(1-c_1)(-1/3) = \psi_1 = 1/3$: $2c_1-(1-c_1) = 1$: $c_1 = 2/3$, $c_2 = 1/3$. So

$$\psi_i = \frac{2}{3}(\frac{2}{3})^i + \frac{1}{3}(\frac{-1}{3})^i = (\frac{2}{3})^{i+1} - (\frac{-1}{3})^{i+1},$$

and

$$X_t = \sum_0^\infty [(\frac{2}{3})^{i+1} - (\frac{-1}{3})^{i+1}]\epsilon_{t-i},$$

giving the moving-average representation, as required.
Autocovariance. Recall the Yule-Walker equations

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2)$$

for $AR(2)$. As before,

$$\rho(k) = a\lambda_1^k + b\lambda_2^k$$

for some constants a, b. Taking $k = 0$ and using $\rho(0) = 1$ gives $a + b = 1$: $b = 1 - a$. So here,

$$\rho(k) = a(2/3)^k + (1-a)(-1/3)^k.$$

Taking $k = 1$ in the Yule-Walker equations gives

$$\rho(1) = \phi_1\rho(0) + \phi_2\rho(-1),$$

which as $\rho(0) = 1$ and $\rho(-1) = \rho(1)$ gives

$$\rho(1) = \phi_1/(1-\phi_2).$$

As here $\phi_1 = 1/3$ and $\phi_2 = 2/9$, this gives $\rho(1) = 3/7$. We can now use this and the above expression for $\rho(k)$ to find a: taking $k = 1$ and equating,

$$\rho(1) = 3/7 = a.(2/3) + (1-a).(-1/3).$$

That is,

$$(\frac{3}{7}+\frac{1}{3}) = a.(\frac{2}{3}+\frac{1}{3}) = a:$$

$a = (9+7)/21 = 16/21$. Thus

$$\rho(k) = \frac{16}{21}(\frac{2}{3})^k + \frac{5}{21}(\frac{-1}{3})^k.$$

Note. For large k, the first term dominates, and

$$\rho^k \sim \frac{16}{21}.(\frac{2}{3})^k \qquad (k \to \infty).$$

Variance. For the variance: square both sides of (2) and take expectations:

$$\gamma_0 = varX_t = E[\sum_{i=0}^{\infty}\psi_i\epsilon_{t-i}.\sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}] = \sum\sum_{i,j=0}^{\infty}\psi_i\psi_j E[\epsilon_{t-i}\epsilon_{t-j}].$$

But $E[\epsilon_{t-i}\epsilon_{t-j}] = 0$ unless $i = j$, when it is $\sigma^2 = var\epsilon_{t-i}$. So

$$\gamma_0 = \sum_{i=0}^{\infty}\psi_i^2 = \sigma^2.\sum_0^{\infty}[(\frac{2}{3})^{i+1} - (\frac{-1}{3})^{i+1}]^2.$$

The constant on the RHS is a sum of geometric series, on squaring out $[...]^2$:

$$\sum_0^{\infty}(4/9)^{i+1} - 2\sum_0^{\infty}(-2/9)^{i+1} + \sum_0^{\infty}(1/9)^{i+1},$$

which sums to

$$\frac{(4/9}{1-(4/9)} - 2.\frac{(-2/9)}{1+(2/9)} + \frac{1/9}{1-(1/9)} = \frac{4}{5} + \frac{4}{11} + \frac{1}{8} = \frac{352+160+55}{440} = \frac{567}{440}.$$

So

$$varX_t = \gamma_0 = \sigma^2.567/440$$

. Similarly,

$$\begin{aligned} \gamma_t = cov(X_t, X_{t-\tau}) &= E[\sum_{i=0}^{\infty}\psi_i\epsilon_{t-i}.\sum_{j=0}^{\infty}\psi_j\epsilon_{t-j} \\ &= \sum\sum_{i,j=0}^{\infty}\psi_i\psi_j E[\epsilon_{t-i}\epsilon_{t-\tau-j}]. \end{aligned}$$

On the RHS, $E[.] = \sigma^2$ if $i = \tau + j$, zero otherwise. So for $\tau \geq 0$,

$$\gamma_\tau = \sigma^2.\sum_{j=0}^{\infty}\psi_{\tau+j}\psi_j = \sigma^2\sum_{j=0}^{\infty}[(\frac{2}{3})^{j+1} - (-\frac{-1}{3})^{j+1}].[(\frac{2}{3})^{\tau+j+1} - (-\frac{-1}{3})^{\tau+j+1}].$$

Again, the constant on RHS is a sum of geometric series,

$$(2/3)^\tau.\frac{4/9}{1-(4/9)} - (-1/3)^\tau.\frac{(-2/9)}{1+(2/9)} - (2/3)^\tau.\frac{(-2/9)}{1+(2/9)} + (1/3)^\tau.\frac{1/9}{1-(1/9)},$$

giving

$$\frac{4}{5}.(\frac{2}{3})^\tau + \frac{2}{11}.(-\frac{1}{3})^\tau + \frac{2}{11}.(\frac{2}{3})^\tau + \frac{1}{8}.(-\frac{1}{3})^\tau = (\frac{2}{3})^\tau.[\frac{4}{5}+\frac{2}{11}] + (-\frac{1}{3})^\tau.[\frac{2}{11}+\frac{1}{8}]:$$

$$\gamma_\tau = \sigma^2.(\frac{54}{55}.(\frac{2}{3})^\tau + \frac{27}{88}.(-\frac{1}{3})^\tau = \frac{\sigma^2}{440}.(8.54.(2/3)^\tau + 5.27.(-1/3)^\tau).$$

So as $\gamma_0 = \sigma^2.567/440$ and $567 = 81.7$,

$$\rho_\tau = \gamma_\tau/\gamma_0 = \frac{8.54}{81.7}.(\frac{2}{3})^\tau + \frac{5.27}{81.7}.(-\frac{1}{3})^\tau,$$

and as $27/81 = 1/3$, $54/81 = 2/3$, we finally get the autocorrelation function of this $AR(2)$ model as

$$\rho_\tau = \frac{16}{21}.(\frac{2}{3})^\tau + \frac{5}{21}.(-\frac{1}{3})^\tau.$$

Check. (a) $\rho_0 = 16/21 + 5/21 = 1$,
(b) We already know $\rho_\tau = a.(2/3)^\tau + b.(-1/3)^\tau$ for some a, b.
Note. For large τ, the first term dominates, and

$$\rho_\tau \sim \frac{16}{21}.(\frac{2}{3})^\tau \qquad (\tau \to \infty):$$

ρ_τ is approximately geometrically decreasing for large τ.
$AR(p)$ processes (continued). We return to the general case. Just as in the $AR(2)$ example above, if the $AR(p)$ process has a moving-average representation

$$X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i},$$

then if $\sigma^2 = var\epsilon_t$,

$$varX_t = \sigma^2.\sum_{i=0}^{\infty} \psi_i^2.$$

The condition

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty$$

(in words: (ψ_i) is *square-summable*, or is *in* L_2) is necessary and sufficient for

(i) $varX_t < \infty$;
(ii) the series $\sum \psi_i \epsilon_{t-i}$ in the moving-average representation to be convergent in mean square – or, in L_2.
So if we interpret convergence in the mean-square sense, $\sum \psi_i^2 < \infty$ is the necessary and sufficient condition (NASC) for the moving-average representation of X_t to *exist*. Since $\sum \psi_i \epsilon_{t-i}$ is (when convergent) stationary (because (ϵ_t) is stationary: if $\sum \psi_i^2 < \infty$, then X_t is stationary. The converse is also true; see Section 5 below.