smprob5.tex

SMF PROBLEMS 5. 18.2.2011

Q1 (*Rank-one matrices*). Show that a matrix $A = (a_{ij})$ has rank 1 iff $A = ab^T$, for column vectors A, b, that is, iff $a_{ij} = b_i c_j$ for some b_i, c_j (then A is called the *tensor product* of the vectors a and $b, a \otimes b$).

Note. The SVD $A = l_1 u_1 v_1^T + \ldots + l_r u_r v_r^T$ thus expresses a matrix A of rank r as a sum of r matrices of rank 1. The SVD is thus also called the rank-one decomposition. If as we may we rank the l_i in decreasing order, $l_1 \geq \ldots l_r > 0$, then for each $k \leq r$

$$A_k := l_1 u_1 v_1^T + \ldots + l_k u_k v_k^T$$

gives an approximation to A as a sum of k rank-one matrices. This gives (in ways that can be made precise using various matrix norms) the best approximation of a rank-r matrix by a rank-k matrix with k < r (the Eckart-Young Theorem). This is often useful in practice.

Q2 (Generalised inverses and SVD). Show that if A has SVD $A = ULV^T$, then $A^- := VL^{-1}U^T$ is a generalised inverse of A.

Q3 (Consistency condition). For A an $n \times n$ matrix and b a column n-vector, write (A, b) for the matrix obtained by adjoining the vector b as the last column of the matrix A. Show that the equation

Ax = b

is consistent (i.e., has at least one solution x) iff A and (A, b) have the same rank. Deduce that there are three cases:

- (i) Unique solution: $|A| \neq 0$;
- (ii) Infinitely many solutions: |A| = 0, r(A) = r((A, b));
- (iii) No solution: |A| = 0, r(A) < r((A, b)).

Q4. Find the eigenvalues, eigenvectors and ranks of the following matrices:

$$A = \begin{pmatrix} a^2 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a^2 + b & a & a \\ a & 1 + b & 1 \\ a & 1 & 1 + b \end{pmatrix}.$$
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