smfprob4.tex

SMF SOLUTIONS 4. 18.2.2011

Q1 (*Product theorem for determinants*). We follow G. BIRKHOFF & S. MAC LANE, *A survey of modern algebra*, rev. ed., Macmillan, 1953, X.2.

A can be diagonalised by pre- and post-multiplication by elementary matrices:

$$A = E_r \ldots E_1 D E^1 \ldots E^s,$$

and for each such E, $|EA| = |E|.|A|$, $|AE| = |A|.|E|$. Then A is non-singular iff all entries of D are non-zero, when D, being diagonal, is itself an elementary matrix, $A = E_1 \ldots E_k$ say. If B is also non-singular, then $B = E'_1 \ldots E'_l$ say, and then by above

$$|AB| = |E_1 \ldots E_k E'_1 \ldots E'_l| = |E_1| \ldots |E_k|.|E'_1| \ldots |E'_l| = |A|.|B|.$$

If B is singular, then $Bx = 0$ for some non-zero vector x, and then $ABx = A.0 = 0$, so AB is singular; similarly (by taking transposes, which does not affect the determinant but switches factors), if A is singular, AB is singular. When either factor is singular, the result holds with both sides 0.

Q2. Recall that, with A_{ij} the cofactor (= signed minor) of a_{ij},

$$|A| = \sum_j a_{ij} A_{ij} = \sum_i a_{ij} A_{ij}$$

for each i (expanding by the ith row) and each j (expanding by the jth column). So expanding by the ith column, $|A| = \sum_k a_{ki} A_{ki}$. For A_i, the ith column is b, with kth component b_k, so $|A_i| = \sum_k b_k A_{ki}$. But the solution of $Ax = b$ is $x = A^{-1}b$, and the inverse matrix has i, j element $A^{-1}_{ij} = A_{ji}/|A|$ (inverse matrix = transposed matrix of cofactors over determinant). So

$$x_i = (A^{-1}b)_i = \sum_k A^{-1}_{ik} b_k = \sum_k A_{ki} b_k/|A| = \sum_k b_k A_{ki}/|A| = |A_i|/|A|.$$

Q3. $x^T Ax = \sum_{ij} a_{ij} x_i x_j$, so by linearity of E, $E[x^T Ax] = \sum_{ij} a_{ij} E[x_i x_j]$. Now $cov(x_i, x_j) = E(x_i x_j) - (Ex_i)(Ex_j)$, so

$$\begin{aligned} E[x^T Ax] &= \sum_{ij} a_{ij}[cov(x_i x_j) + Ex_i.Ex_j] \\ &= \sum_{ij} a_{ij} cov(x_i x_j) + \sum_{ij} a_{ij}.Ex_i.Ex_j. \end{aligned}$$

The second term on the right is $Ex^T Ax$. For the first, note that

$$trace(AB) = \sum_i (AB)_{ii} = \sum_{ij} a_{ij} b_{ji} = \sum_{ij} a_{ij} b_{ij},$$

if B is symmetric. But covariance matrices are symmetric, so the first term on the right is $trace(Avar(x))$, as required.

Q4. (i) $P^2 = A(A^T A)^{-1} A^T . A(A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$; $(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$.
(ii) Recall that $tr(A + B) = tr(A) + tr(B)$, and that $tr(AB) = tr(BA)$. So

$$trace(I - AC^{-1} A^T) = trace(I) - trace(AC^{-1} A^T).$$

But $trace(I) = n$ (as here I is the $n \times n$ identity matrix), and as $trace(AB) = trace(BA)$, $trace(AC^{-1} A^T) = trace(C^{-1} A^T A) = trace(I) = p$, as here I is the $p \times p$ identity matrix. So $trace(I - AC^{-1} A^T) = n - p$.
(iii) If λ is an eigenvalue of B, with eigenvector x, $Bx = \lambda x$ with $x \neq 0$. Then

$$B^2 x = B(Bx) = B(\lambda x) = \lambda (Bx) = \lambda (\lambda x) = \lambda^2 x,$$

so λ^2 is an eigenvalue of B^2 (always true – i.e., does not need idempotence). So

$$\lambda x = Bx = B^2 x = \ldots = \lambda^2 x,$$

and as $x \neq 0$, $\lambda = \lambda^2$, $\lambda(\lambda - 1) = 0$: $\lambda = 0$ or 1.

The trace is the sum of the eigenvalues, which is r if there are r eigenvalues, i.e. when the rank is r. So trace = rank.
(iv) Because P is a projection of rank r, it has r eigenvalues 1 and the rest 0. We can diagonalise it by an orthogonal transformation to a diagonal matrix with r 1s on the diagonal, followed by 0s. So the quadratic form $x^T Px$ can be reduced to a sum of r squares of standard normal variates, $y_1, \ldots, y_r$. These are independent $N(0, \sigma^2)$ (if $y = Ox$ with O orthogonal and the x_i iid $N(0, 1)$, then the y_i are also iid $N(0, 1)$: for, the joint density of the x_i involves only $\|x\|$, which is preserved under an orthogonal transformation). So $x^T Px = y_1^2 + \ldots y_r^2$ is σ^2 times a $\chi^2(r)$-distributed random variable.

NHB