

SMF SOLUTIONS 5. 18.2.2011

Q1 (*Rank-one matrices*). If C is the zero matrix, it has rank 0 – a trivial case, which we exclude.

If C has rank one, the range of C is one-dimensional (this is one of several equivalent definitions of rank). If the domain and range of C have bases e_i, f_j , Ce_i is non-zero for some i (or C would be zero) – w.l.o.g., $Ce_1 = \sum_j c_{1j}f_j \neq 0$. Write $b_j := c_{1j}$: $Ce_1 = \sum_j b_j f_j \neq 0$. As the range of C is one-dimensional, for each i , Ce_i is a multiple $a_i Ce_1$ of Ce_1 : $Ce_i = \sum_j a_i b_j f_j$. This says that the linear transformation represented by C has matrix $C = (c_{ij}) = (a_i b_j)$ w.r.t. the bases e_i and f_j .

Conversely, if $C = (a_i b_j)$ is not the zero matrix: at least one $a_i b_j \neq 0$; w.l.o.g., by re-ordering rows and columns, take $a_1, b_1 \neq 0$. Then column j is the multiple b_j/b_1 of column 1, and row i is the multiple a_i/a_1 of row 1. So C has column-rank 1 (only 1 linearly independent column), and row-rank 1, so has rank 1.

Q2. $AA^+A = ULV^TVL^{-1}U^TULV^T = ULL^{-1}LV^T = ULV^T = A$, as $U^TU = I$ and $V^TV = I$.

Q3. With $A = (a_1, \dots, a_n)$ a row of its columns and $x = (x_1, \dots, x_n)^T$, $Ax = x_1 a_1 + \dots + x_n a_n$ is a linear combination of the columns of A . So $Ax = b$ can have a solution iff b is a linear combination of the columns of A , i.e. iff adding B to the space spanned by the columns does not increase its dimension, i.e. iff $r((A, b)) = r(A)$.

We know $|A| \neq 0$ is the condition for uniqueness, which is (i). With non-uniqueness but consistency, $Ax_1 = b$ and $Ax_2 = b$ for $x_1 \neq x_2$. Then $A(x_1 - x_2) = 0$, so $A \cdot c(x_1 - x_2) = 0$ for all c , so $A(x_1 + c(x_1 - x_2)) = b$ for all c , giving infinitely many solutions, which is (ii). With inconsistency, there are no solutions, giving (iii).

Q4. With \mathbf{a} the column-vector $(a, 1, 1)^T$, $A = \mathbf{a}\mathbf{a}^T$. So A has rank 1, and as it is 3×3 , it has one non-zero eigenvalues and two 0 eigenvalues.

$$A\mathbf{a} = \mathbf{a}\mathbf{a}^T\mathbf{a} = (a^2 + 2)\mathbf{a},$$

as $\mathbf{a}^T \mathbf{a} = a^2 + 2$. This says that A has eigenvalue $a^2 + 2$ with eigenvector \mathbf{a} . The other two eigenvalues are 0, with eigenvectors \mathbf{x} , \mathbf{y} say.

$B = A + bI$. So

$$B\mathbf{a} = A\mathbf{a} + bI\mathbf{a} = (a^2 + 2)\mathbf{a} + b\mathbf{a} = (a^2 + b + 2)\mathbf{a}.$$

This says that B has eigenvalue $a^2 + b + 2$ with eigenvector \mathbf{a} .

As $A\mathbf{x} = 0$, $B\mathbf{x} = A\mathbf{x} + b\mathbf{x} = b\mathbf{x}$, which says that B has eigenvalue b with eigenvector \mathbf{x} . Similarly, B has eigenvalue b with eigenvector \mathbf{y} . So: B has eigenvalues $a^2 + b + 2, b, b$; its eigenvectors are the same as those of A ; its rank is 3, unless $b = 0$, when its rank is 1.

NHB