smfprob11.tex

SMF PROBLEMS 11. 13.6.2012

Q1 (*Product theorem for determinants*). Show that $|AB| = |A|.|B|$.
Note. This is done in lectures one way; this is to show an alternative approach, via elementary row and column operations, as in the classic text G. BIRKHOFF and S. MacLANE, *A survey of modern algebra*, rev. ed., Macmillan, 1953.

Q2 (*Cramer's Rule*). If $Ax = b$ with A (square and) non-singular, the solution vector x has ith component $x_i = |A_i|/|A|$, where A_i is the matrix obtained from A by replacing its ith column by the RHS b.

Q3 (*Trace Formula*). Show that for x a random vector, with mean Ex and covariance matrix $var(x)$, the mean of the quadratic form $x^T Ax$ is

$$E[x^T Ax] = trace(A.var(x)) + (Ex)^T.A.Ex.$$

Q4. Recall that a matrix A is a projection iff it is idempotent ($A^2 = A$), and that in regression, the design matrix A is $n \times p$, with $p << n$. The *projection matrix* P (also known as the *hat matrix* H) is $P := A(A^T A)^{-1} A$.
(i) Show that P, $I - P$ are projections.
(ii) Show that they have traces $tr(P) = p$, $tr(I - P) = n - p$.
(iii) Show that for an idempotent matrix, the eigenvalues are 0 or 1, and its trace is its rank.

Q5. If P is a projection of rank r and x_i are independent $N(0, \sigma^2)$, show that the quadratic form $x^T Px$ is σ^2 times a $\chi^2(r)$-distributed random variable.

NHB