smfprob12.tex

SMF PROBLEMS 12. 15.6.2012

Q1 (*Rank-one matrices*). Show that a matrix $A = (a_{ij})$ has rank 1 iff $A = ab^T$, for column vectors A, b, that is, iff $a_{ij} = b_i c_j$ for some b_i, c_j (then A is called the *tensor product* of the vectors a and b, $a \otimes b$).

Note. The SVD $A = l_1 u_1 v_1^T + \ldots + l_r u_r v_r^T$ thus expresses a matrix A of rank r as a sum of r matrices of rank 1. The SVD is thus also called the *rank-one decomposition*. If as we may we rank the l_i in decreasing order, $l_1 \geq \ldots l_r > 0$, then for each $k \leq r$

$$A_k := l_1 u_1 v_1^T + \ldots + l_k u_k v_k^T$$

gives an approximation to A as a sum of k rank-one matrices. This gives (in ways that can be made precise using various matrix norms) the *best approximation* of a rank-r matrix by a rank-k matrix with $k < r$ (the *Eckart-Young Theorem*). This is often useful in practice.

Q2 (*Generalised inverses and SVD*). Show that if A has SVD $A = ULV^T$, then $A^- := VL^{-1}U^T$ is a generalised inverse of A.

Q3 (*Consistency condition*). For A an $n \times n$ matrix and b a column n-vector, write (A, b) for the matrix obtained by adjoining the vector b as the last column of the matrix A. Show that the equation

$$Ax = b$$

is consistent (i.e., has at least one solution x) iff A and (A, b) have the same rank. Deduce that there are three cases:
(i) *Unique solution*: $|A| \neq 0$;
(ii) *Infinitely many solutions*: $|A| = 0$, $r(A) = r((A, b))$;
(iii) *No solution*: $|A| = 0$, $r(A) < r((A, b))$.

Q4. Find the eigenvalues, eigenvectors and ranks of the following matrices:

$$A = \begin{pmatrix} a^2 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a^2 + b & a & a \\ a & 1 + b & 1 \\ a & 1 & 1 + b \end{pmatrix}.$$

NHB