smfprob2.tex

SMF PROBLEMS 2. 11.5.2012

Q1 (*Bernoulli distribution* $B(p)$). The *Bernoulli distribution* with parameter $p \in (0,1)$ has mass p on 1, mass $1-p$ on 0 ('tossing a biased coin'). Show that
(i) $\bar{x} := \frac{1}{n}\sum_1^n x_i$ is the MLE for p;
(ii) $\bar{x}$, or $\sum x_i$, is minimal sufficient for p.

Q2 (*Minimal sufficiency for the multivariate normal*). The *multivariate normal distribution* (in d dimensions) $N(\mu, \Sigma$ (μ a d-vector, Σ an $d\times d$ symmetric positive definite matrix) has density (V.3, D7)

$$f(\mathbf{x}) := \frac{1}{(2\pi)^{\frac{1}{2}d}|\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\}.$$

With $V := \Sigma^{-1}$, we shall show (V.4, D8) that the likelihood

$$L = (2\pi)^{-nd/2}|\Sigma|^{-n/2}\exp\{-\frac{1}{2}\sum_1^n(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)\}$$

$$= (2\pi)^{-nd/2}|V|^{n/2}\exp\{-\frac{1}{2}n\ trace(VS) - (\bar{x}-\mu)^T V(\bar{x}-\mu)\}.$$

where the *trace* of a square matrix is the sum of its diagonal elements, and $barx$, S are the sample mean and sample covariance matrix,

$$\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i, \qquad S = S_x := \frac{1}{n}\sum_{i=1}^n (x_i-\bar{x})^T(x_i-\bar{x}).$$

Show that $(\bar{x}, S)$ is minimal sufficient for (μ, Σ).

Q3 *Rayleigh distribution*. The *Rayleigh distribution* with parameter $v > 0$ is the distribution on $(0,\infty)$ with density

$$f(x) := (x/v)\exp\{-\frac{1}{2}x^2/v^2\} \qquad (x > 0)$$

(it arises in Statistical Mechanics in the modelling of velocity, whence the 'v'). Show that

(i) this is a density;
(ii) $\frac{1}{2}X^2$ has mean v and variance v^2;
(iii) the information per reading is $1/v^2$.

Deduce that
$$\hat{v} := \frac{1}{2}.\frac{1}{n}\sum_{1}^{n} X_i^2$$
is unbiased and efficient for v.

NHB