smfsoln10.tex

**SMF SOLUTIONS 10. 13.6.2012**

Q1. (i) The roots $\lambda_1, \dots, \lambda_p$ of the polynomial $\lambda^p - \phi_1 \lambda^{p-1} - \dots - \phi_{p-1}\lambda - \phi_p$ should lie inside the unit disk.
(ii) Multiply $(*)$ by $X_{t-k}$ for $k \geq 0$ and take expectations: $E[X_t] = 0$, and

$$\gamma_k = cov(X_t, X_{t-k}) = E[X_t X_{t-k}] = \phi_1 E[X_{t-1}X_{t-k}] + \dots + \phi_p E[X_{t-p}X_{t-k}] + E[\epsilon_t X_{t-k}].$$

As $\epsilon_t$ has mean 0 and is independent of $X_{t-k}$, this gives

$$\gamma_k = \phi_1 \gamma_{k-1} + \dots + \phi_p \gamma_{k-p}.$$

Divide by $\gamma_0$:

$$\rho_k = \phi_1 \rho_{k-1} + \dots + \phi_p \rho_{k-p}.$$

(iii) General solution $\rho_k = c_1 \lambda_1^k + \dots + c_p \lambda_p^k$, $c_i$ constants.

Q2. (i)

$$\gamma_0 = var(X_0) = var(X_t) = E[X_t^2] = E[(\epsilon + \theta\epsilon_{t-1})(\epsilon + \theta\epsilon_{t-1})] = \sigma^2(1 + \theta^2),$$

as $E[\epsilon_t^2] = E[\epsilon_{t-1}^2] = \sigma^2$, $E[\epsilon_t \epsilon_{t-1}] = 0$. (ii)

$$\gamma_1 = E[X_t X_{t-1}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2})] = \sigma^2\theta,$$

$$\gamma_2 = E[X_t X_{t-2}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-2} + \theta\epsilon_{t-3})] = 0,$$

and similarly $\gamma_k = 0$ for $k \geq 2$.
(iii) $\rho_k = \gamma_k/\gamma_0$. So

$$\rho_0 = 1, \qquad \rho_{\pm 1} = \theta/(1 + \theta^2), \qquad \rho_k = 0 \quad \text{otherwise}.$$

Q3. (i) $(X_t)$ is $ARMA(2, 1)$.
(ii) $X_t - X_{t-1} + \frac{1}{4}X_{t-2} = \epsilon_t + \frac{1}{2}\epsilon_{t-1}$; with $B$ the backward shift,

$$\phi(B)X_t = \theta(B)\epsilon_t,$$

where $\phi(\lambda) = 1 - \lambda + \frac{1}{4}\lambda^2 = (1 - \frac{1}{2}\lambda)^2$, with a repeated root at $\lambda = 2$,
$\theta(\lambda) = 1 + \frac{1}{2}\lambda$, root $\lambda = -2$.

All roots are outside the unit disk in the complex $\lambda$-plane, so $(X_t)$ is stationary and invertible.

Q4. For an orthogonal matrix $B$, $B^T B = I$. So by the Product Theorem for Determinants, $|B^T||B| = |I| = 1$, $|B|^2 = 1$, $|B| = 1$ (the case $|B| = -1$ corresponds to reversing the direction of an axis – anti-orthogonality – and we are about to take the modulus anyway). So the Jacobian of the transformation $X \mapsto Y$ is mod det $\partial y_i / \partial x_j$ = mod det $b_{ij} = 1$ ($y_i = \sum_j b_{ij} x_j$). Orthogonal transformations preserve lengths of vectors, so

$$\|y\| = \|x\| : \qquad \sum y_i^2 = \sum x_i^2.$$

The joint density of the $x_i$ is

$$f(x_1, \dots x_n) = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2} \sum x_i^2\}.$$

So by the Jacobian formula, the joint density of the $y_i$ is

$$f(y_1, \dots y_n) = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2} \sum y_i^2\}$$

(for background on such uses of the Jacobian formula when changing variables in Statistics, see e.g. [BF], 2.2). This says that $Y =_d X$: $y_1, \dots, y_n$ are iid $N(0, 1)$.

NHB