smfprob4.tex

SMF SOLUTIONS 4. 23.5.2012

Q1. As in Problems 3 Q3, the likelihood for the Cauchy location family does not have any non-trivial factorisation. So by the Fisher-Neyman Theorem, there are no non-trivial sufficient statistics.

Q2. Under H_0,

$$L(\mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{1}{2}m+\frac{1}{2}n}\sigma^{m+n}} \exp\{-\frac{1}{2}(\sum_1^m (X_i - \mu)^2 + \sum_1^n (Y_j - \mu)^2))/\sigma^2\},$$

$$\ell = const - (m+n)\log\sigma - \frac{1}{2}(\sum_1^m (X_i - \mu)^2 + \sum_1^n (Y_j - \mu)^2))/\sigma^2.$$

$$\partial/\partial\sigma = 0: \quad -\frac{m+n}{\sigma} + \frac{1}{\sigma^3}(\Sigma_1 + \Sigma_2) = 0: \quad \sigma^2 = (\sum_1^m (X_i - \mu)^2 + \sum_1^n (Y_j - \mu)^2))/(m+n).$$

$$\partial/\partial\mu = 0: \qquad \sum_1^m (X_i - \mu) + \sum_1^n (Y_j - \mu) = 0:$$

$$\mu = \frac{1}{m+n}(\sum_1^m X_i + \sum_1^n Y_j) = \frac{m\bar{X} + n\bar{Y}}{m+n} = \hat{\mu},$$

say. Substitute above: at the maximum,

$$\sigma^2 = \frac{1}{m+n}(\sum_1^m (X_i - \hat{\mu})^2 + \sum_1^n (Y_j - \hat{\mu})^2), = \hat{\sigma}_0^2,$$

say. So

$$\sup_{H_0} L(\mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{1}{2}m+\frac{1}{2}n}\hat{\sigma}_0^{m+n}} \exp\{-(m+n)\},$$

as $\hat{\sigma}_0^2$ *cancels* in the exponent.

Under H_1, we obtain as above

$$\hat{\mu}_1 = \bar{X}, \quad \hat{\mu}_2 = \bar{Y}, \quad \hat{\sigma}^2 = \frac{mS_x^2 + nS_y^2}{m+n} = \hat{\sigma}_1^2,$$

say. Then

$$\sup_{H_1} L(\mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{1}{2}m+\frac{1}{2}n}\hat{\sigma}_1^{m+n}} \exp\{-(m+n)\},$$

as $\hat\sigma_1^2$ *cancels* in the exponent. The likelihood ratio is thus

$$\lambda = (\sup_{H_0} L)/(\sup_{H_1} L) = (\hat\sigma_1/\hat\sigma_0)^{m+n}.$$

So

$$\lambda^{2/(m+n)} = (\hat\sigma_1/\hat\sigma_0)^2 = \frac{\sum_1^m (X_i - \bar X)^2 + \sum_1^n (Y_j - \bar Y)^2}{\sum_1^m (X_i - \frac{m\bar X + n\bar Y}{m+n})^2 + \sum_1^n (Y_j - \frac{m\bar X + n\bar Y}{m+n})^2}.$$

The first term in the denominator is

$$\sum_1^m [(X_i - \bar X) + (\bar X - \frac{m\bar X + n\bar Y}{m+n})]^2 = \sum_1^m (X_i - \bar X)^2 + m.\frac{n^2}{(m+n)^2}(\bar X - \bar Y)^2$$

(the cross-terms sum to 0, the third term simplifies), and similarly for the second term. This gives

$$\lambda^{2/(m+n)} = \frac{\sum_1^m (X_i - \bar X)^2 + \sum_1^n (Y_j - \bar Y)^2}{\sum_1^m (X_i - \bar X)^2 + \sum_1^n (Y_j - \bar Y)^2 + \frac{mn}{m+n}(\bar X - \bar Y)^2}.$$

So

$$\lambda^{-2/(m+n)} = 1 + \frac{\frac{mn}{m+n}(\bar X - \bar Y)^2}{\sum_1^m (X_i - \bar X)^2 + \sum_1^n (Y_j - \bar Y)^2} = 1 + t^2/(m+n-2)$$

(see Problems 4). The LR test is: reject if λ too small, which by above is t^2 too big, $|t|$ too big – the t-test, as required.

Q3. The likelihood is

$$L = \prod_1^n [I(x_i \in (0,\theta))/\theta] = \theta^{-n} I(\max < \theta), \qquad \max := \max(x_1, \ldots, x_n).$$

So the posterior density is proportional to (prior times likelihood)

$$\lambda e^{-\lambda\theta}/\theta^n, \qquad (\theta > \max).$$

NHB