smfsoln6.tex

SMF SOLUTIONS 6. 30.5.2012

Q1. (i) The likelihood is

$$L = \prod_1^n \theta^{-1} I(x_i \in (0,\theta)) = \theta^{-n} I(\theta > \max), \qquad \max := \max(x_1, \ldots, x_n).$$

To maximise this, one minimises θ, subject to the constraint $\theta > \max$. So the MLE is $\hat{\theta} = \max$.
(ii) By Fisher-Neyman, for each n $\max(x_1, \ldots, x_n)$ is a sufficient statistic.
(iii) Posterior is proportional to prior times likelihood, so

$$f(\theta | x_1, \ldots, x_n) \propto \lambda e^{-\lambda\theta} . \theta^{-n} \qquad (\theta > \max(x_1, \ldots, x_n)).$$

Q2. The relevant densities are

$$f(y|u) = const. \exp\{-\frac{1}{2}(y - X\beta - Zu)^T R^{-1} (y - X\beta - Zu)\}, \quad f(u) = const. \exp\{-\frac{1}{2} u^T D^{-1} u\}.$$

By Bayes' Theorem,

$$f(u|y) = f(y|u) f(u)/f(y) = f(u,y)/f(y).$$

But the denominator $f(y)$ is just an 'integration constant', so

$$f(u|y) \propto \exp\{-\frac{1}{2}[(y - X\beta - Zu)^T R^{-1} (y - X\beta - Zu) + u^T D^{-1} u]\}.$$

This has the functional form of a multivariate normal (in u). So it *is* a multivariate normal, and we can find *which* one by picking out the matrix Σ in the $u^T \Sigma^{-1} u$ term, and then the vector μ in the $u^T \Sigma^{-1} \mu$ term. We can read off first

$$\Sigma^{-1} = Z^T R^{-1} Z + D^{-1} : \qquad \Sigma = (Z^T R^{-1} Z + D^{-1})^{-1},$$

and then

$$Z^T R^{-1}(y - X\beta) = \Sigma^{-1}\mu : \quad \mu = \Sigma Z^T R^{-1} (y - X\beta) = (Z^T R^{-1} Z + D^{-1})^{-1} Z^T R^{-1} (y - X\beta).$$

NHB