smfsoln8.tex

SMF SOLUTIONS 8. 6.6.2012

Q1. (i) For $n = 1$, the mean is 1, because a $\chi^2(1)$ is the square of a standard normal, and a standard normal has mean 0 and variance 1. The variance is 2, because the fourth moment of a standard normal X is 3, and

$$var(X^2) = E[(X^2)^2] - [E(X^2)]^2 = 3 - 1 = 2.$$

For general n, the mean is n because means add, and the variance is $2n$ because variances add over independent summands.
(ii) For X standard normal, the MGF of its square X^2 is

$$M(t) := \int e^{tx^2} \phi(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{tx^2}.e^{-\frac{1}{2}x^2} dx.$$

We see that the integral converges only for $t < \frac{1}{2}$, when it is $1/\sqrt{(1-2t)}$:

$$M(t) = 1/\sqrt{1-2t} \qquad (t < \frac{1}{2}) \qquad \text{for } X \ N(0,1).$$

Now when X, Y are independent, the MGF of their sum is the product of their MGFs. For, e^{tX}, e^{tY} are independent, and the mean of an independent product is the product of the means. Combining these, the MGF of a $\chi^2(n)$ is given by

$$M(t) = 1/(1-2t)^{\frac{1}{2}n} \qquad (t < \frac{1}{2}) \qquad \text{for } X \ \chi^2(n).$$

(iii) First, $f(.)$ is a density, as it is non-negative, and integrates to 1:

$$\begin{aligned} \int f(x)dx &= \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}. \int_0^\infty e^{tx}.x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x)dx \\ &= \frac{1}{\Gamma(\frac{1}{2}n)}. \int_0^\infty u^{\frac{1}{2}n-1} \exp(-u)du \qquad (u := \frac{1}{2}x) \\ &= 1, \end{aligned}$$

by definition of the Gamma function. Its MGF is

$$M(t) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}. \int_0^\infty e^{tx}.x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x)dx$$

$$= \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} . \int_0^\infty x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x(1-2t))dx.$$

Substitute $u := x(1-2t)$ in the integral. One obtains

$$M(t) = (1-2t)^{-\frac{1}{2}n} . \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} . \int_0^\infty u^{\frac{1}{2}n-1} e^{-u} du = (1-2t)^{-\frac{1}{2}n},$$

by definition of the Gamma function. //

Q2. (i) $A^T A$ is symmetric, so $P = A(A^T A)^{-1} A^T$ is symmetric. $P^2 = A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$.
(ii) $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$, so $I - P$ is a (symmetric) projection.
(a) $tr(A+B) = tr(A) + tr(B)$ follows as the trace is additive from its definition.
$tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij} b_{ji}$, and this is $tr(BA)$ on interchanging the dummy duffices i and j.
(b) $tr(P) = tr(A(A^T A)^{-1} A^T) = tr(A^T A (A^T A)^{-1}) = tr(I_p) = p$, as A is $n \times p$, so $A^T A$ is $p \times p$.
$tr(I-P) = tr(I_n) - tr(P) = n - p$, as $P = A(A^T A)^{-1} A^T$ is $n \times n$.

NHB