smfsoln1.tex

SMF SOLUTIONS 1. 11.5.2012

Q1.

$$log f(X,\mu) = -\frac{1}{2}(X-\mu)^2/\sigma^2 - \frac{1}{2}\log 2\pi - \log\sigma.$$

$$\partial \log f(X,\mu)/\partial\mu = (X-\mu)/\sigma^2.$$

The information per reading is

$$E[\partial \log f/\partial\mu)^2] = E[(X-\mu)^2/\sigma^4] = \sigma^2/\sigma^4 = 1/\sigma^2.$$

So the information in the whole sample is $I = n/\sigma^2$, so the CR bound is $1/I = \sigma^2/n$. But $\bar{X}$ is unbiased (mean μ), with variance σ^2/n, the CR bound. So $\bar{X}$ is efficient for μ.

Q2. Write $v := \sigma^2$.

$$\log f = const - \frac{1}{2}\log v - \frac{1}{2}(X-\mu)^2/v,$$

$$\partial \log f/\partial v = -\frac{1}{2v} + \frac{(X-\mu)^2}{2v^2} = \frac{1}{2v^2}[(X-\mu)^2 - v].$$

The information per reading is

$$E[(\partial \log f/\partial v)^2] = \frac{1}{4v^4}[E\{(X-\mu)^4\} - 2vE[(X-\mu)^2] + v^2].$$

Now $N(\mu,\sigma^2)$ has MGF $M(t) := E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, so $X-\mu$ has MGF $\exp\{\frac{1}{2}\sigma^2 t^2\} = \exp\{-\frac{1}{2}vt^2\}$,

$$M(t) = 1 + \frac{1}{2}vt^2 + \frac{1}{8}v^2t^4 + \ldots = \sum \mu_k t^k/k!, \qquad \mu_k = E[(X-\mu)^k].$$

$k=4$: $\mu_4/4! = \mu_4/24 = v^2/8$: $\mu_4 := E[(X-\mu)^4] = 3v^2$, $\mu_2 = varX = \sigma^2 = v$. So the information per reading is

$$\frac{3v^2 - 2v.v + v^2}{4v^4} = \frac{1}{2v^2},$$

and the CR bound is $2v^2/n$. But $\frac{1}{n}\sum_1^n (X_i - \mu)^2$ is unbiased (mean $\frac{1}{n}\sum_1^n \sigma^2 = \sigma^2 = v$), with variance

$$\frac{1}{n^2}.n\ var(X-\mu)^2 = \frac{1}{n}\{E[(X-\mu)^4] - (E(X-\mu)^2])^2\} = \frac{1}{n}(3v^2 - v^2) = 2v^2/n,$$

the CR bound. So $\frac{1}{n}\sum_1^n (X_i - \mu)^2$ is an efficient estimator for σ^2.

Q3. We know S_u^2 is unbiased for $\sigma^2 = v$. The CR bound is $2v^2/n$, as in Q2. Now (with S^2 the (biased) sample variance) $nS^2/\sigma^2 \sim \chi^2(n-1)$, which has mean $n-1$ (the number of degrees of freedom, df) and variance $2(n-1)$. So $S_u^2 = (n/(n-1))S^2$ has (mean μ and) variance

$$var\ S_u^2 = \frac{n^2}{(n-1)^2} var\ S^2 = \frac{n^2}{(n-1)^2}.\frac{\sigma^4}{n^2}.2(n-1) = \frac{2\sigma^4}{n-1} = \frac{2v^2}{n-1} = \frac{n}{n-1}.2v^2/n.$$

So the efficiency is $(n-1)/n = 1 - 1/n \to 1$: S_u^2 is asymptotically efficient for $v = \sigma^2$.

Q3 *Student t-distribution.* The Student t-distribution with r degrees of freedom (df), $t(r)$, has density

$$f_{t(r)}(x) = \frac{\Gamma(\frac{1}{2}r + \frac{1}{2})}{\sqrt{\pi r}\Gamma(\frac{1}{2}r)}.\Big(1 + \frac{x^2}{r}\Big)^{-\frac{1}{2}(r+1)}.$$

Now by Stirling's formula, $\Gamma(r) \sim \sqrt{2\pi} e^{-x} x^{x+\frac{1}{2}}$ as $x \to \infty$. Using this and $(1 + x/n)^n \to e^x$ gives

$$\Gamma(x+a) \sim x^a \Gamma(x) \qquad (x \to \infty).$$

So the ratio of Gammas $\sim \sqrt{\frac{1}{2}r}$, while the bracket $\sim e^{-\frac{1}{2}x^2}$. Combining,

$$f_{t(n)}(x) \to e^{-\frac{1}{2}x^2}/\sqrt{2\pi} \qquad (n \to \infty),$$

the density $\phi(x)$ of $N(0,1)$: $t(n) \to N(0,1)$.

Alternatively,

$$(\bar{X} - \mu)\sqrt{n-1}/S \sim t(n-1), \qquad (\bar{X} - \mu)\sqrt{n}/\sigma \sim \Phi = N(0,1).$$

By LLN, $S \to \sigma$, and $\sqrt{n-1} \sim \sqrt{n}$, so $t(n-1) \to N(0,1)$, i.e. $t(n) \to N(0,1)$.

NHB