smfsoln11(13).tex

SMF SOLUTIONS 12. 14.6.2013

Q1.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} AM - BD^{-1}CM & -AMBD^{-1} + BD^{-1} + BD^{-1}CMBD^{-1} \\ CM - CM & -CMBD^{-1} + I + CMBD^{-1} \end{pmatrix}.$$

The $(1,1)$ element is I, from the definition of M. For the $(1,2)$ entry, the first and third terms combine to give $-BD^{-1}$, again by definition of M, so the $(2,1)$ element is 0. The $(2,1)$ element is clearly 0. In the $(2,2)$ element, the first and third terms cancel, so the $(2,2)$ element is I. //

Q2.

$$K_{11} = M := \Sigma_{11} - \Sigma_{12}\Sigma_{22}\Sigma_{21},$$

$$K_{12} = -M\Sigma_{12}\Sigma_{22}, \quad \text{so} \quad K_{11}^{-1}K_{12} = -\Sigma_{12}\Sigma_{22}^{-1}.$$

By the last theorem of IV.6 D6, the covariance matrix of $x_1|x_2$ is the partial covariance matrix $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. With Σ the partitioned matrix in Q3, this is M^{-1}, in the notation of Q3. By Edgeworth's theorem, this identifies the concentration matrix K_{11} of $x_1|x_2$ as $K_{11} = M$:

$$K_{11} = M; \qquad M := (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}.$$

If x_1 is a 2-vector, Σ_{11}, K_{11} are 2×2 matrices. Now (III.1)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a 2×2 matrix A is diagonal iff its inverse A^{-1} is diagonal. For a Gaussian vector, two components are independent iff they are uncorrelated, i.e. iff their (2×2) covariance matrix is diagonal. So: for a 2-vector x_1, with $x^T = (x_1^T, x_2^T)$: the components of x_1 are conditionally independent given x_2 (i.e., given all the other components of x) iff K_{11} is diagonal, i.e. iff $k_{12} = 0$ in an obvious notation. Similarly for k_{ij} for any $i \neq j$. So: components x_i, x_j of a random vector $x \sim N(\mu, \Sigma)$ are conditionally independent given the other components iff $k_{ij} = 0$.

NHB