smfsoln2.tex

SMF SOLUTIONS 2. 16.5.2012

Q1. (i) $f(x) = p^x(1-p)^{1-x}$ (this takes the required values p at 1 and $1-p$ at 0). So

$$L = p^{\sum x_i}(1-p)^{n-\sum x_i} = p^{n\bar{x}}(1-p)^{n(1-\bar{x})}, \quad \ell = n\bar{x}\log p + n(1-\bar{x})\log(1-p).$$

So

$$\ell' = \frac{n\bar{x}}{p} - \frac{n(1-\bar{x})}{1-p},$$

which is 0 iff

$$\frac{\bar{x}}{p} = \frac{1-\bar{x}}{1-p}, \qquad \bar{x} - p\bar{x} = p - p\bar{x}, \qquad p = \bar{x}.$$

This is indeed a maximum (as one can check), so $\bar{x}$ is the MLE for p.
(ii)

$$\frac{L(x)}{L(y)} = \Big(\frac{p}{1-p}\Big)^{\sum x_i - \sum y_i}.$$

This is independent of p iff $\sum x_i = \sum y_i$, i.e. iff $\bar{x} = \bar{y}$. so by Lehmann-Scheffé, $\bar{x}$ is minimal sufficient for p.

Q2. In an obvious notatiion,

$$\ell_x = const - \frac{1}{2}n\log|\Sigma| - \frac{1}{2}n\ trace(VS_x) - (\bar{x}-\mu)^T V(\bar{x}-\mu)$$

$$= const - \frac{1}{2}n\log|\Sigma| - \frac{1}{2}n\ trace(VS_x) - \bar{x}^T V\bar{x} - 2\mu^T V\bar{x} + \mu^T V\mu$$

(the two cross-terms are scalars, so are their own transposes, so we can combine them), and similarly for ℓ_y. Subtract:

$$\ell_x - \ell_y = -\frac{1}{2}n\ trace[V(S_x - S_y)] - [\bar{x}^T V\bar{x} - \bar{y}V\bar{y}] - 2\mu^T V(\bar{x}-\bar{y}).$$

This is independent of the parameters μ and Σ (or V) iff

$$\bar{x} = \bar{y}, \qquad S_x = S_y.$$

So by the Lehmann-Scheffé theorem, $(\bar{x}, S_x)$ is minimal sufficient for (μ, Σ).

Q3 *Rayleigh distribution.* (i) Putting $u := \frac{1}{2}x^2/v$, $du = (x/v)dx$, so

$$\int f(x)dx = \int_0^\infty e^{-u}du = 1,$$

so f is indeed a density.
(ii)

$$E[\frac{1}{2}X^2] = \int_0^\infty \frac{1}{2}x^2.\exp\{-\frac{1}{2}x^2/v\}.xdx/v = v\int_0^\infty (\frac{1}{2}x^2/v)\exp\{-\frac{1}{2}x^2/v\}d(\frac{1}{2}x^2/v)$$

$$= v\int_0^\infty te^{-t}dt = v$$

(integrating by parts, or by $\Gamma(2) = 1! = 1$). So the mean is v.
The same substitution gives

$$E[(\frac{1}{2}X^2)^2] = v^2\int_0^\infty t^2e^{-t}dt = 2v^2$$

(integrate by parts, or use $\Gamma(3) = 2! = 2$). So

$$var(\frac{1}{2}X^2) = E[(\frac{1}{2}X^2)^2] - (E[\frac{1}{2}X^2)]^2) = 2v^2 - v^2 = v^2 :$$

$\frac{1}{2}X^2$ has variance v^2.
(iii)

$$\ell = \log f = \log x - \log v - \frac{1}{2}x^2/v.$$

So the score function is

$$s = \ell' = -\frac{1}{v} + \frac{1}{2}.\frac{x^2}{v^2} = \frac{x^2 - 2v}{2v^2}.$$

So the information per reading is

$$I = E[s^2] = E\Big[(\frac{x^2 - 2v}{2v^2})\Big] = \frac{1}{4v^4}[E(X^4) - 4vE(X^2) + 4v^2]$$

$$= \frac{1}{4v^4}[8v^2 - 4v.2v + 4v^2] = \frac{4v^2}{4v^4} = 1/v^2.$$

So the CR lower bound is v^2/n. But by (i) $\hat{v}$ is unbiased for v, and by (ii) it attains the CR bound, so is efficient for v.

NHB