smfsoln9.tex

SMF SOLUTIONS 7. 31.5.2013

Q1. From the model equation

$$y_i = \sum_{j=1}^{p} a_{ij}\beta_j + \epsilon_i, \qquad \epsilon_i \quad iid \quad N(0,\sigma^2),$$

the likelihood is

$$\begin{aligned} L &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}}.\prod_{i=1}^{n} \exp\{-\frac{1}{2}(y_i - \sum_{j=1}^{p} a_{ij}\beta_j)^2/\sigma^2\} \\ &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}}.\exp\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - \sum_{j=1}^{p} a_{ij}\beta_j)^2/\sigma^2\}, \end{aligned}$$

and the log-likelihood is

$$\ell := \log L = const - n\log\sigma - \frac{1}{2}[\sum_{i=1}^{n}(y_i - \sum_{j=1}^{p} a_{ij}\beta_j)^2]/\sigma^2. \qquad (*)$$

We use Fisher's Method of Maximum, and maximise with respect to β_r in $(*)$ – or equivalently, the Method of Least Squares to minimise [...]: $\partial\ell/\partial\beta_r = 0$ gives

$$\sum_{i=1}^{n} a_{ir}(y_i - \sum_{j=1}^{p} a_{ij}\beta_j) = 0 \qquad (r = 1,\ldots,p),$$

or

$$\sum_{j=1}^{p}(\sum_{i=1}^{n} a_{ir}a_{ij})\beta_j = \sum_{i=1}^{n} a_{ir}y_i.$$

Write $C = (c_{ij})$ for the $p\times p$ matrix

$$C := A^T A,$$

which we note is *symmetric*: $C^T = C$. Then

$$c_{ij} = \sum_{k=1}^{n}(A^T)_{ik}A_{kj} = \sum_{k=1}^{n} a_{ki}a_{kj}.$$

So this says

$$\sum_{j=1}^{p} c_{rj}\beta_j = \sum_{i=1}^{n} a_{ir}y_i = \sum_{i=1}^{n}(A^T)_{ri}y_i.$$

In matrix notation, this is

$$(C\beta)_r = (A^T y)_r \qquad (r = 1,\ldots,p),$$

or combining,

$$C\beta = A^T y, \qquad C := A^T A. \tag{NE}$$

These are the *normal equations.* As A ($n \times p$, with $p << n$) has full rank, A has rank p, so $C := A^T A$ has rank p, so is non-singular. So the normal equations have solution

$$\hat{\beta} = C^{-1} A^T y = (A^T A)^{-1} A^T y.$$

Multiplying both sides by A,

$$Py = A(A^T A)^{-1} A^T y = A\hat{\beta}.$$

Q2. $\partial \ell / \partial \sigma = 0$ gives $-n/\sigma + [...]/\sigma^3 = 0$, $\sigma^2 = [...]/n$. At the maximum, $\beta = \hat{\beta}$, so $[...] = SSE$, giving $\hat{\sigma}^2 = SSE/n$.

$$\begin{aligned} SSE &= (y - A\hat{\beta})^T (y - A\hat{\beta}) \\ &= y^T (I - P)^T (I - P) y \quad \text{(by Q1)} \\ &= y^T (I - P) y \qquad (P^T = P,\ P^2 = P \text{ as } P \text{ is a symmetric projection}). \end{aligned}$$

Q3. For an orthogonal matrix B, $B^T B = I$. So by the Product Theorem for Determinants, $|B^T||B| = |I| = 1$, $|B|^2 = 1$, $|B| = 1$ (the case $|B| = -1$ corresponds to reversing the direction of an axis – anti-orthogonality – and we are about to take the modulus anyway). So the Jacobian of the transformation $X \mapsto Y$ is mod det $\partial y_i / \partial x_j$ = mod det $b_{ij} = 1$ ($y_i = \sum_j b_{ij} x_j$). Orthogonal transformations preserve lengths of vectors, so

$$\|y\| = \|x\| : \qquad \sum y_i^2 = \sum x_i^2.$$

The joint density of the x_i is

$$f(x_1, \ldots x_n) = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2} \sum x_i^2\}.$$

So by the Jacobian formula, the joint density of the y_i is

$$f(y_1, \ldots y_n) = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2} \sum y_i^2\}$$

(for background on such uses of the Jacobian formula when changing variables in Statistics, see e.g. [BF], 2.2). This says that $Y =_d X$: $y_1, \ldots, y_n$ are iid $N(0, 1)$.

NHB