smfsoln8(13).tex

SMF SOLUTIONS 8. 5.6.2013

Q1. (i) The roots $\lambda_1, \ldots, \lambda_p$ of the polynomial $\lambda^p - \phi_1\lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p$ should lie inside the unit disk.

(ii) Multiply $(*)$ by X_{t-k} for $k \geq 0$ and take expectations: $E[X_t] = 0$, and

$$\gamma_k = cov(X_t, X_{t-k}) = E[X_t X_{t-k}] = \phi_1 E[X_{t-1}X_{t-k}] + \ldots + \phi_p E[X_{t-p}X_{t-k}] + E[\epsilon_t X_{t-k}].$$

As ϵ_t has mean 0 and is independent of X_{t-k}, this gives

$$\gamma_k = \phi_1\gamma_{k-1} + \ldots + \phi_p\gamma_{k-p}.$$

Divide by γ_0:

$$\rho_k = \phi_1\rho_{k-1} + \ldots + \phi_p\rho_{k-p}.$$

(iii) General solution $\rho_k = c_1\lambda_1^k + \ldots + c_p\lambda_p^k$, c_i constants.

Q2. (i)

$$\gamma_0 = var(X_0) = var(X_t) = E[X_t^2] = E[(\epsilon + \theta\epsilon_{t-1})(\epsilon + \theta\epsilon_{t-1})] = \sigma^2(1 + \theta^2),$$

as $E[\epsilon_t^2] = E[\epsilon_{t-1}^2] = \sigma^2$, $E[\epsilon_t\epsilon_{t-1}] = 0$.

(ii)

$$\gamma_1 = E[X_t X_{t-1}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-1} + \theta\epsilon_{t-2})] = \sigma^2\theta,$$
$$\gamma_2 = E[X_t X_{t-2}] = E[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-2} + \theta\epsilon_{t-3})] = 0,$$

and similarly $\gamma_k = 0$ for $k \geq 2$.

(iii) $\rho_k = \gamma_k/\gamma_0$. So

$$\rho_0 = 1, \qquad \rho_{\pm 1} = \theta/(1 + \theta^2), \qquad \rho_k = 0 \quad \text{otherwise.}$$

Q3. (i) (X_t) is $ARMA(2, 1)$.

(ii) $X_t - X_{t-1} + \frac{1}{4}X_{t-2} = \epsilon_t + \frac{1}{2}\epsilon_{t-1}$; with B the backward shift,

$$\phi(B)X_t = \theta(B)\epsilon_t,$$

where $\phi(\lambda) = 1 - \lambda + \frac{1}{4}\lambda^2 = (1 - \frac{1}{2}\lambda)^2$, with a repeated root at $\lambda = 2$,
$\theta(\lambda) = 1 + \frac{1}{2}\lambda$, root $\lambda = -2$.

All roots are outside the unit disk in the complex λ-plane, so (X_t) is stationary and invertible.

NHB