smfprob1(13a)

SMF PROBLEMS 1. 18.10.2013

Q1. In a normal model $N(\mu, \sigma^2)$, show that $\bar{X}$ is efficient for μ.

Q2. In $N(\mu, \sigma^2)$ with μ known, show that $\frac{1}{n}\sum_1^n (X_i - \mu)^2$ is efficient for $v := \sigma^2$.

Q3. In $N(\mu, \sigma^2)$ with σ the parameter of interest but μ unknown (so a nuisance parameter), show that the unbiased sample variance

$$S_u^2 := \frac{1}{n-1}\sum_1^n (X_i - \bar{X})^2$$

is asymptotically efficient for $v := \sigma^2$, with efficiency $1 - 1/n \to 1$.

Q4 (*Minimal sufficiency for the multivariate normal*). The *multivariate normal distribution* (in d dimensions) $N(\mu, \Sigma)$ (μ a d-vector, Σ an $d \times d$ symmetric positive definite matrix) has density (Edgeworth's Theorem: IV.3, D5)

$$f(\mathbf{x}) := \frac{1}{(2\pi)^{\frac{1}{2}d}|\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \mu)\}.$$

With $V := \Sigma^{-1}$, we shall show (IV.4, D6) that the likelihood

$$L = (2\pi)^{-nd/2}|\Sigma|^{-n/2} \exp\{-\frac{1}{2}\sum_1^n (x_i - \mu)^T \Sigma^{-1}(x_i - \mu)\}$$

$$= (2\pi)^{-nd/2}|V|^{n/2} \exp\{-\frac{1}{2}n\ trace(VS) - (\bar{x} - \mu)^T V(\bar{x} - \mu)\}.$$

where the *trace* of a square matrix is the sum of its diagonal elements, and $\bar{x}$, S are the sample mean and sample covariance matrix,

$$\bar{x} := \frac{1}{n}\sum_{i=1}^n x_i, \qquad S = S_x := \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^T (x_i - \bar{x}).$$

Show that $(\bar{x}, S)$ is minimal sufficient for (μ, Σ).

NHB