smfprob2(13a).tex

SMF PROBLEMS 2. 25.10.2013

Q1 (*Conditions for equality in the Cramér-Rao (Information, CR) Inequality*).
(i) Show that in the Cauchy-Schwarz Inequality

$$(\int fg)^2 \leq (\int f^2)(\int g^2),$$

equality holds iff there is a linear relationship between f and g:

$$af + bg = 0$$

for some constants a, b.
(ii) Deduce that we have equality in CR iff

$$u = a\ell' + b$$

for some a, b. Find b.
(iii) Observing that the constant a above may depend on the parameter θ, and that when we integrate ℓ' to get ℓ, L the constant of integration may depend on the data $\mathbf{X}$, show that equality holds iff L has the form

$$L = \exp\{\alpha(\theta)u(\mathbf{X}) + \beta(\theta) + k(\mathbf{X})\}.$$

Such likelihoods form the *exponential family* – roughly, the families for which one can do parameter estimation satisfactorily.
(iv) Show that $u(\mathbf{X})$ is (a) sufficient for θ; (b) minimal sufficient for θ.

Q2 (*Symmetric exponential location family*). Here

$$f(x) = \frac{1}{2}\exp\{|x - \theta|\}.$$

(i) Show that

$$\ell = const - \sum |x_i - \theta|.$$

Show that this is maximised where θ is the *median* of the sample, $Med = Med(x_1, \ldots, x_n)$, and deduce that this is the MLE:

$$\hat{\mu} = Med.$$

(ii) Show that the information per reading is 1 (use $I = \int (\partial \log f / \partial \theta)^2 f$).

We quote that the sample median Med is asymptotically normal with mean the (population) median med and variance $1/(4nf(med)^2)$.

(iii) Show that Med is asymptotically normal, unbiased and efficient.

Q3 (*Cauchy location family*). The Cauchy location family is defined by

$$f(x;\mu) = \frac{1}{\pi(1 + (x-\mu)^2)}.$$

(i) Show that this does not belong to the exponential family (it is a standard example of this!)

(ii) Show that the MLE has asymptotic variance

$$var(\hat{\mu}) \sim 2/n$$

and efficiency $8/\pi^2$ ($\sim 81\%$). You may quote that

$$I := \int_{-\infty}^{\infty} \frac{x^2}{[1+x^2]^3} dx = \frac{1}{2}.$$

NHB